TASK 3.3.2: MAKING APPLICATION PROBLEMS RICHER

Solutions

I. A large holiday complex contains 100 units. If the rent is $300 per week, all units will be rented. However, for each additional $20 charged in rent, 5 additional units will become vacant.

a. Write an equation that relates rent, *C* (the cost of renting a unit for a week), and the occupancy, *N* (the number of units rented).

$$N = 100 - 5\left(\frac{C - 3000}{20}\right)$$

$$N = 100 - \frac{C}{4} + 75$$

$$N = 175 - \frac{C}{4}$$

This can be found by making a table of values and then fitting a linear function to those values, either "by hand" or using the linear regression option in the STAT menu of the calculator.

b. Find the equation relating total weekly rental income, *R*, with weekly rental rates, *C*.

$$R = C \times N$$

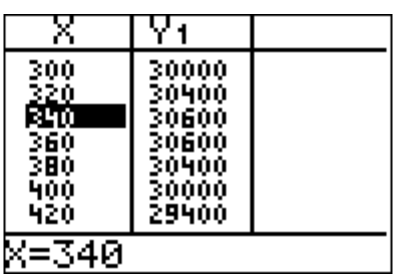
$$R = C\left(175 - \frac{C}{4}\right)$$

$$R = -\frac{1}{4}C^2 + 175C$$

c. What are a reasonable domain and range for *R* as a function of *C*?

First, clearly C needs to be positive. However, it doesn't make sense to charge less than $300 per week for the rentals, so the domain of R "starts" are 300. The rental income will be 0 when the linear function giving the occupancy is also 0, i.e. when C = 700. Thus the domain that makes sense for this situation is [300,700].

To find the range, enter the function R = -.25C^2 *+175C into the calculator and use the table to find a reasonable range.*

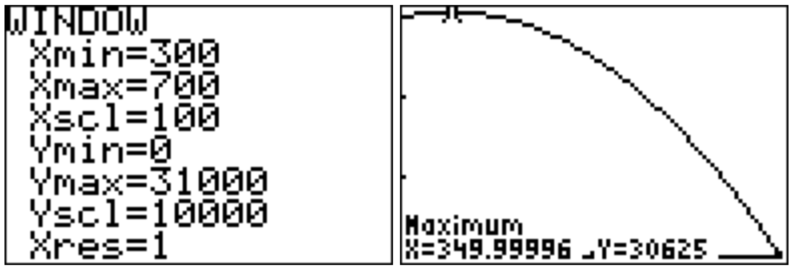


X	Y1	
300	30000	
320	30400	
340	30600	
360	30600	
380	30400	
400	30000	
420	29400	

X=340

It appears that the function reaches a peak between C =340 and C = 360. So a good guess for the range might be [0, 31,000].

d. At what weekly rent per unit is the total weekly rent maximized? Justify your answer in two different ways (algebraically, graphically, or tabular).



Using the maximum option on the calculator, the maximum occurs when C = 350. Optionally, one could complete the square in the function to put it into vertex form, and read the vertex (maximum) right off of that.

$$R = -\frac{1}{4}C^2 + 175C$$

$$= -\frac{1}{4}(C - 350)^2 + 30625$$

So the vertex is (350, 30625). Thus the maximum rental income occurs when $350 is charged per week per unit for rent.

e. What is the total maximum weekly rental income?
From the work above, we saw that the maximum rental income is $30625 per week.

II. Below are three standard textbook quadratic application problems. For each situation, expand the problem so that the road to the answer takes a path that makes as many connections as possible. Be sure to require multiple representations.

Sample examples, extension questions, and answers are below.

1. The product of two numbers whose sum is 144 is to be made as large as possible. What are the two numbers?

   a. Write an equation that relates the sum of the two numbers, x and y.
   x + y =144

   b. Write an equation that relates the product, P, of the two numbers, x and y.
   P = xy

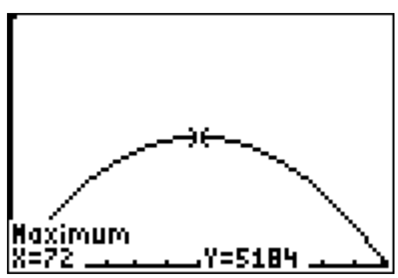
c. We want to make the product of the two numbers, P, as big as possible. How can we use the first equation to write P so that it is a <u>function</u> of x? Why do we want to do this anyway?
Solve the first equation for y as a function of x, then substitute that in the second equation. We get $P = x(144 - x) = -x^2 + 144x$.

We want to do this so we can use our knowledge of functions to analyze P. If P is a function of two variables, we don't have as many tools at this stage of mathematics.

d. What are a reasonable domain and range for P?
Because we are not told that either x or y needs to be positive, any value of x is valid. So the domain is all real numbers.

To find a reasonable range, examine a table of values for the function, or mentally multiply two possible values of x and y to get a sample P value. For instance, if $x = 100$*, then* $y = 44$*, and then* $P = 4400$*. So a reasonable guess for the range would be [0,10,000]. Graph it with this window and then refine the range if it is too big or too small.*

e. Using at least two different methods, find the maximum value for the product of x and y.
Maximum value of P is 5184



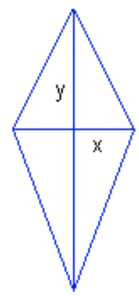
You can find the maximum algebraically by completing the square to put the function into vertex form. Alternately you could examine a table of values.

X	Y1	
68	5168	
70	5180	
72	5184	
74	5180	
76	5168	
78	5148	
80	5120	

X=72

f. What are the values of x and y that correspond to this maximum value of P?
$x = 72, y = 144 - 72 = 72$

2. A two-stick, diamond shaped kite is to be designed with a total stick length of 120 inches. What dimensions will yield the maximum surface area?
    a. What is the area of a kite whose frame is made with two sticks of lengths x and y that are at right angles to each other?



The area of each triangle that makes up the right or left half of the quadrilateral is $\frac{1}{2}\times\left(\frac{x}{2}\right)\times y$. *Since there are two of these triangles, the total area is*

$$A = 2\left(\frac{1}{2}\times\left(\frac{x}{2}\right)\times y\right)$$

$$A = \frac{xy}{2}$$

    b. Write an equation that relates the sum of x and y.
    $x + y = 120$

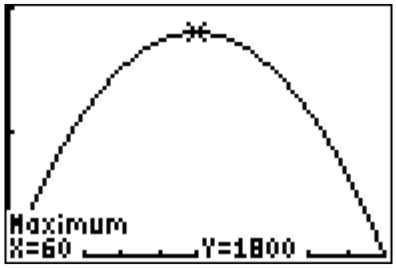
    c. To maximize the area A, we need to write it as a function of one variable. (Why?) Choose either x or y, and write A as a function of that variable.
    $A = .5x(120 - x) = .5\, y(120 - y)$

    d. What are a reasonable domain and range for the function A?
    The length of one stick can be no smaller than 0 and no larger than 120. So the domain is [0, 120]. The range can be estimated using a table of values. A good approximation to the range is [0, 2000].

    e. What are the units of A?
    square inches

    f. Using at least two different methods, find the maximum surface area for the kite.

Maximum surface area is 1800in^2.



The vertex form of the quadratic function for A is $A = -.5(x - 60)^2 + 1800$.

g. What are the lengths of the cross pieces for the kite of maximum surface area?
$x = 60$ and $y = 120 - 60 = 60$.

h. Invent a real world situation where you would have a need to solve this problem.
Suppose you have a length of balsa wood that is exactly 120 inches long (borrowed your friend who likes to build model airplanes) and as much nylon fabric as you could want (borrowed from your other friend who likes to build parachutes). You are going to build a diamond shaped kite and you want your kite to catch as much air as possible, i.e. its surface area should be as large as possible. How should you cut your balsa wood to make the frame so that the surface area is maximized?

3. Design a box with a rectangular cross-section and a length of 36 inches so that it will have the maximum carrying capacity possible. You want the girth (the perimeter of the cross-section) to also be 36 inches. What dimensions will create the biggest box?

   a. What does it mean to make the biggest box? Biggest height, most area, widest?
   The biggest box is the one that will hold the most, i.e. the one with the most volume.

   b. Let l be the length of the box, w be the width, and h be the height. Write a formula for the volume of the box, V, in terms of l, w, and h.
   $V = lwh$

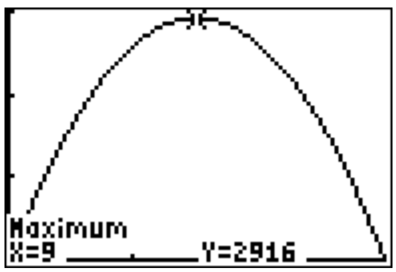
   c. Do you know any values for l, w, or h?
   $l = 36$.

   d. Write an equation that gives a relationship between w and h.
   $2w + 2h = 36$

e. Use the equation you just found to write V as a function of w. (Why would you want to do this?)
$V = 36wh = 36w(18 - w) = -36w^2 + 648w$

f. What are a reasonable domain and range for V?
w must be at least zero and can be no larger than 18 (otherwise it would be impossible to build a box with a cross-sectional perimeter of 36. So the domain is [0,18]. A good estimate for the range is [0, 3000]. This upper bound comes from computing a trial value of V. For instance, if w = 10, then h = 8 and V = 36 x 10 x 8; V = 2880.

g. Using at least two different methods, find the maximum volume of the box.



The maximum volume is $2916in^3$.

X	Y1	
7	2772	
8	2880	
9	2916	
10	2880	
11	2772	
12	2592	
13	2340	

X=9

h. What are the dimensions of this box?
l = 36, w = 9, h = 9. So the cross-section is a square.

Math notes
This task provides a good way of connecting algebra and geometry.

Teaching notes
This task gives a great opportunity to have the participants practice the art of leading a task by providing their own scaffolding questions for a problem. Ask each group to work on developing one problem, then ask one group to present each problem to the class by leading the rest of the participants through the problem using their guiding questions. The groups can either put their work on chart paper or use a projector.

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e) What is the total maximum weekly rental income?

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