

TASK 2.3.5: FLYING THROUGH THE ST. LOUIS GATEWAY ARCH**Solutions**

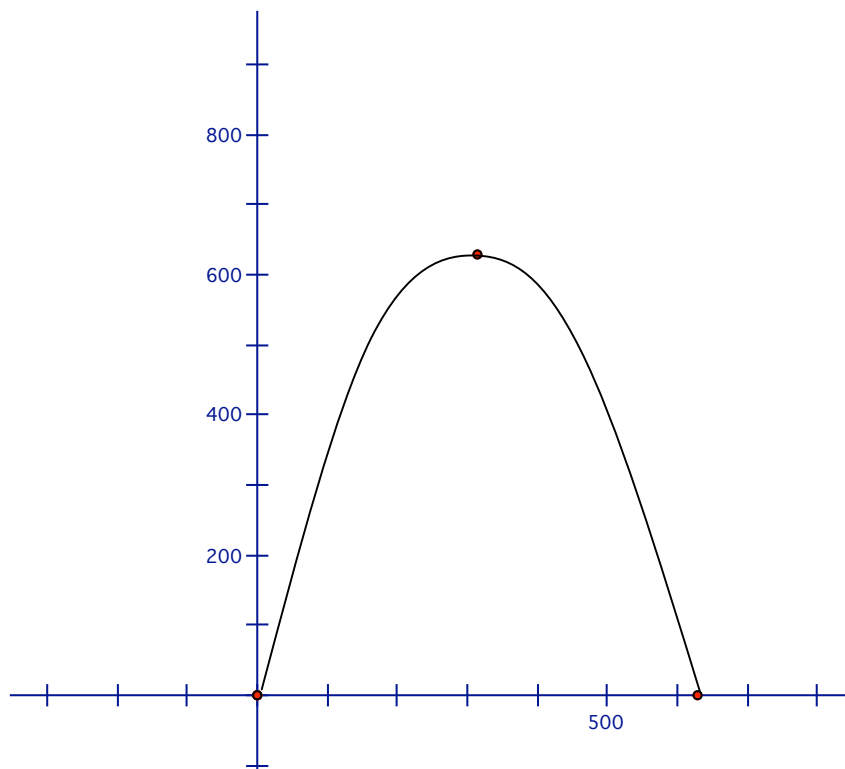
On a recent visit to St. Louis, Missouri, Anna visited the St. Louis Gateway Arch. On the tour, she learned that the arch is 630 feet tall and the legs of the arch are 630 ft. apart. Anna, being on spring break from her Algebra II class, looks at the arch and immediately sees a parabola. In a fervent effort to analyze this amazing curve, she realizes that if she only knew three points, then she could determine the equation of the parabola.

1. Based on the information she learned in the tour, could Anna come up with three coordinates? If so, what might they be? Are these the only possible coordinates?

There are several ways to orient the coordinate grid that would give different sets of coordinates. Two common solutions are:

$(0, 0)$, $(630, 0)$ and $(315, 630)$;
 $(-315, 0)$, $(0, 630)$ and $(315, 0)$

2. Suppose you orient a coordinate grid with the bottom of one leg of the arch at the origin and the rest of the arch lying in the first quadrant so that the other leg is also on the x-axis. Sketch the arch in the first quadrant below and label as many coordinates as you can on the arch.



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3. Using the general form of a quadratic function, $y = Ax^2 + Bx + C$, and the three coordinates labeled above, write a system of three equations to solve for the three unknowns, A, B and C.

$$\begin{aligned} C &= 0 \\ 6302A + 630B + C &= 0 \\ 3152A + 315B + C &= 630 \end{aligned}$$

4. Write the above system of three equations and three unknowns as a matrix equation.

$$\begin{bmatrix} 0 & 0 & 1 \\ 396900 & 630 & 1 \\ 99225 & 315 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 630 \end{bmatrix}$$

5. Use inverse matrices to solve for A, B, and C and write the quadratic equation that would model the St. Louis Arch and sketch its graph.

- What do we know about matrix multiplication that is different from real number multiplication? Matrix multiplication is not commutative.
- Do matrices have multiplicative inverses that function the same way real number multiplicative inverses do? Not all matrices have inverses. Only square matrices can have an inverse. If a square matrix does have an inverse, it functions the same way as a multiplicative inverse in the real number system. That is, if matrix A has an inverse, A^{-1} , then $A \cdot A^{-1} = I = A^{-1} \cdot A$, where I is the identity matrix.
- Does every square matrix have an inverse? Give examples. No, for example,

the matrix $\begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}$ does not have an inverse. If we try to find the inverse with the calculator, we get an error message. Notice that the second row is a factor of the first row. If you think of this matrix as a coefficient matrix for a system of linear equations, the values represent the slopes of the equation. For example a system that might use this coefficient matrix would be $2x + 5y = K$ and $6x + 15y = L$. Even though we don't know what K and L are, we can say that these two lines would be parallel and the system would have no solution.

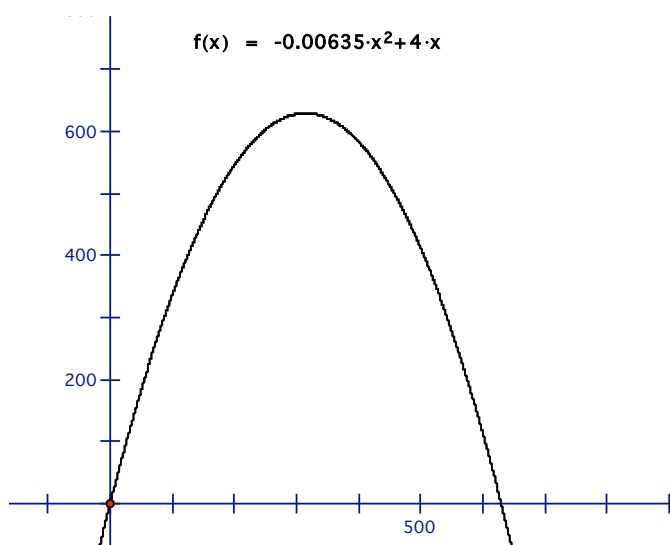
To use inverses to solve the matrix equation in question 4, we would multiply both sides of the equation by the inverse of the 3x3 matrix. On the left side, this would leave the identity matrix, and on the right side we could calculate our solution matrix.

$$\begin{bmatrix} 0 & 0 & 1 \\ 396900 & 630 & 1 \\ 99225 & 315 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 396900 & 630 & 1 \\ 99225 & 315 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 396900 & 630 & 1 \\ 99225 & 315 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 0 \\ 630 \end{bmatrix}$$

Using a graphing calculator to perform the matrix operations, we get

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -.00635 \\ 4 \\ 0 \end{bmatrix}$$

So, a quadratic equation that would model the St. Louis Arch is $y = -.00635x^2 + 4x$



6. Is there another way to orient the coordinate axes that might make finding the quadratic equation easier? Choose another orientation and find the quadratic equations using the new coordinate system. How are the two quadratic equations related?

One possible way would be to center the legs of the arch on either side of the origin. The three coordinate would then be $(-315, 0)$, $(0, 630)$ and $(315, 0)$. Using our knowledge of transformations of the parent function $y = x^2$ we know that this function is $y = -Ax^2 + 630$. In order to find the value of A , we can plug in one of the points on the x -axis.

$$0 = -A(315)^2 + 630$$

- Before you solve for A in the equation above, what do you think that the coefficient of x^2 will be? It should be the same as in the last quadratic model because the two graphs should only differ by a translation. The value of A affects the opening of the arch.

Solving for A gives use the equation $y = -.00635x^2 + 630$.

7. Although the tour guide told Anna that the airspace around the St. Louis Arch is restricted by the FAA, Anna still wonders if it would be possible for an airplane to fly through the arch. Anna's father is a small aircraft pilot and tells Anna that to safely avoid the high rise buildings around the arch, a plane would have to maintain an altitude of at least 300 feet and that the plane would need at least 75 feet on each side for error. What would be the largest wingspan an aircraft could have to safely fly through the arch? Explain your solution.

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In order to answer this question, we must set our model equation equal to 300.

- Does it matter which model equation we use? No, they describe equivalent curves. So, we must find the intersection of $y=300$ and $y=-.00635x^2+630$. We can do this in a number of ways. We can look at the graph and trace the intersection point, zooming in until we have enough accuracy. We can look at the table of values and change our table set to find the solutions with the desired accuracy. Or we can set the two equations equal to each other and solve algebraically.

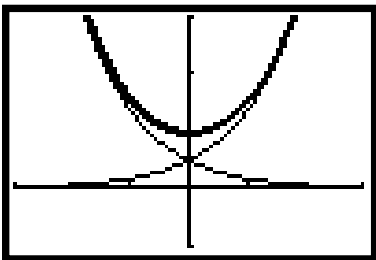
Using the model center around the y-axis, the solutions lie around $x=\pm 228$. This leaves us a span of 456 feet. If we need to take off 75 feet from both sides, the largest wingspan that will fit is $(456-150)=306$ feet.

8. The formula actually used in constructing the St. Louis Arch is displayed on the inside of the arch. It is the formula for a catenary curve, which is the shape a free hanging chain takes when held at both ends. Mathematically, the function that models such a curve is hyperbolic cosine. The formula used for the St. Louis Arch is $y=68.8 \cosh(.01x-1)$. Use your graphing calculator to graph this function. (If you have trouble finding the cosh function, use the catalog feature of your calculator.) Is this graph close to the graph of either of the quadratic functions you found? Can you transform either of the functions (the quadratic or the hyperbolic cosine) in a way that preserves size to show that they model close to the same physical structure?

The catenary curve described by the formula is a reflection and a translation of the curve we modeled with the quadratic. In order to see this, we can manipulate only the parameters that will not change the size of our arch. The catenary curve can be transformed to $y=-68.8\cosh(.01x-3.15)+700$. This graph and the first quadratic model we created are very close together. Some discrepancies occur because of where the tour guide's measurements are taken. The tour guide tells Anna that the legs are 630 feet apart. This measurement is taken from the outside edge of both legs. A graphic of the actual arch with the published catenary curve overlaid shows the catenary curve does not follow the outer edge of the arch, but instead represents an internal structure. Still, the two curves are very close.

9. The hyperbolic functions are a special class of exponential functions which have many parallels to the trigonometric functions: Hyperbolic cosine, cosh, is defined

as $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Below is a graph of $y = \frac{e^x}{2}$ and $y = \frac{e^{-x}}{2}$ along with the graph of $y = \cosh(x)$ in bold.



Can you think of other objects or situations we often think of as parabolas that might in fact make more sense to model with a catenary curve?

Telephone wires hanging between two poles, a hammock.

Math notes

In this activity, students plot points on a coordinate grid to model the St. Louis Gateway Arch. They look at various ways to place a coordinate system and find different quadratic equations to model the arch based on the different coordinate layouts. They also set up a system of equations to find the coefficients of a quadratic. Students use inverse matrices to solve for the coefficients of the quadratic equation. Students then use their understanding of transformations to compare the two quadratic equations they have to model the arch. Finally, students learn the true function used to construct the arch and compare it to their models.

Teaching notes

Read through the opening paragraph with students and answer question 1 together. Set up question 2 as a class and then have students work in groups on questions 3-7. Discuss their solutions as a class and then discuss question 8 together.

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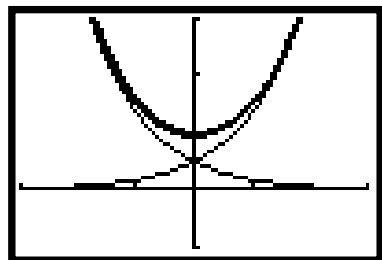
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