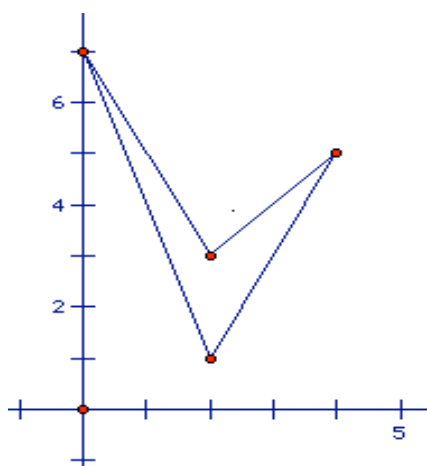


TASK 2.3.3: TRANSFORMATIONS USING MATRICES**Solutions**

1. Let matrix A below represent a set of ordered pairs, where the first row of the matrix represents the x-values and the second row of the matrix represents the y-values. On graph paper, sketch the polygon formed by connecting each of the coordinates in the order they are listed in the matrix. (Make sure to close the polygon by connecting the last coordinate to the first.)

$$A = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 1 & 7 & 3 & 5 \end{bmatrix}$$



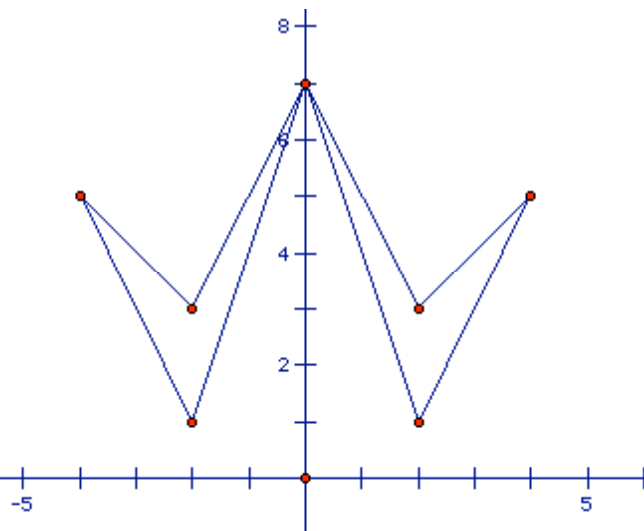
2. Enter matrix A into your graphing calculator. Then enter matrix B, below, into your calculator.

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Find the matrix product, $B \times A$. The result will be another set of ordered pairs. On the same graph paper used in question 1, sketch the polygon formed by connecting the coordinates in matrix $B \times A$. How are the two polygons related?

$$B \times A = \begin{bmatrix} -2 & 0 & -2 & -4 \\ 1 & 7 & 3 & 5 \end{bmatrix}$$

The two polygons are reflections of each other across the y-axis.



4. Test your hypothesis by creating several other polygon matrices at your table and finding the product of your new matrices with matrix B. Then plot the results. Record your matrices and the resulting products below. Use the MATPLOT graphing calculator program to help expedite this process. To use this program, matrix A is always the ordered pair matrix and matrix B is the transforming matrix.

Participants should be asked to create at least three polygon matrices on which to test their hypothesis. The resulting matrices and graphs should support their hypothesis.

5. Algebraically, how do we define a reflection across the y-axis?

$$(x, y) \rightarrow (\underline{\quad -x \quad}, \underline{\quad y \quad})$$

6. Find the product of the two matrices below. How does this product relate to the definition in question 5? How does this relate to your hypothesis from question 4?

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

The result is consistent with the hypothesis in question 4. That is, each x-coordinate is replaced by the opposite of the coordinate which produces a reflection across the y-axis.

7. Can you change matrix B so that, when multiplied by a matrix A, the polygon will be reflected across the x-axis? What is the new matrix B?

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

8. How is a rotation about the origin of 180° related to reflections across the x- and y-axes? Can you use this information to find a matrix, $RO, 180^\circ$, that would rotate an object around the origin by 180° ?

A rotation about the origin of 180° is the same as a composition of a reflection about the x-axis and a reflection about the y-axis. So, the product of the two reflection matrices described above would yield a matrix that would transform an object by a rotation of 180° about the origin. So,

$$\text{matrix } RO, 180^\circ = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

9. Find transformation matrices for the following:

$$RO, 90^\circ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$RO, 270^\circ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\text{RO, } 360^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

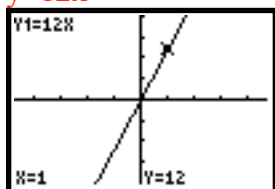
$$\text{RO, } \theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$\text{Dilation, DO, } 3 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Reflect and Apply: Determining Polynomials Solutions

1. How many points determine a line? **Two**
2. Find the equation of the line that passes through the points (0, 0) and (1, 12). Graph the line on your graphing calculator.

$$y=12x$$



3. How many non-collinear points are required to determine a quadratic equation? **Three**
4. Find the quadratic equation that passes through the points (0, 0), (1, 12) and (3, 6). (Set up a system of 3 equations and 3 unknowns. Then use matrices to solve for the unknowns.) Add this graph to the graph above.

The formula for a general quadratic equation is $y=ax^2+bx+c$. We have three values for x and y . So we can create a system of equations to solve for a , b , and c . The system is:

$$\begin{cases} a(0)^2 + b(0) + c = 0 \\ a(1)^2 + b(1) + c = 12 \\ a(3)^2 + b(3) + c = 6 \end{cases}$$

. We can rewrite this system as a matrix equation as follows.

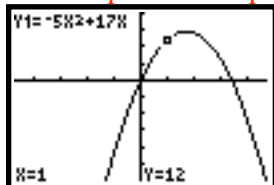
$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 6 \end{bmatrix}$$

. Using a graphing calculator and inverse matrix operations, we can solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 12 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ 17 \\ 0 \end{bmatrix}$$

for a, b, and c.

So our quadratic equation is $y = -5x^2 + 17x$.

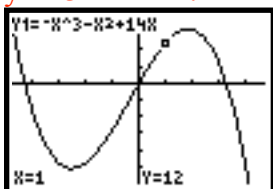


5. How many non-collinear points are required to determine a cubic equation? **Four**
6. Find the cubic equation that passes through the points (0, 0), (1, 12), (3, 6) and (-1, -14). Add this graph to the graph above.

We can set up a matrix equation that is generated using the general cubic equation $y = ax^3 + bx^2 + cx + d$ and the x and y coordinates given by the points.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 6 \\ -14 \end{bmatrix}$$

When we solve this equation, we get $a = -1$, $b = -1$, $c = 14$ and $d = 0$. So our cubic equation is $y = -x^3 - x^2 + 14x$.



Teaching notes

In this Task, we focus upon transformations in the context of matrix operations. After the initial setup in questions 1 and 2, participants are asked to carry out matrix multiplication and determine the effect that multiplying by matrix B has on the original set of points (matrix A). In question 4, participants should produce at least three examples that support their hypothesis regarding the effect of matrix B. Take time before the groups start working with the calculator program to discuss what the program does.

The first major steps in the program transfer the data in our matrices into lists. Why must the matrices be transposed? Why don't we just start with a 4x2 matrix instead of a 2x4 matrix? The matrices have to be transposed because the calculator feature `Matr ► list()` puts each column of a

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matrix into a list. We need each row to be in a list so that we can plot L1 vs. L2 as ordered pairs. We cannot simply write our ordered pair matrix as a 4x2 matrix because we need to multiply our coordinate matrix by a 2x2 transformation matrix.

After some class discussion, participants can work on question 9 for homework and the “Reflect and Apply: Defining Polynomials” task. At the beginning of the next meeting, have participants get into groups and discuss their conclusions to question 9. As they are working, encourage them to make a list of ordered pairs for each transformation and analyze the change in x and y.

What does a rotation of 360° do to the pre-image? The pre-image is directly on top of the image. What matrix doesn't change the entries of the matrix it is multiplied by? The identity matrix.

When looking for the general rotation matrix, what do you notice about the values of each entry in all of the rotation matrices? What function(s) have these values at 0° , 90° , 180° , 270° ? The values are all 0 and ± 1 . Sine and cosine have these values at 90° intervals. So a general rotation

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

matrix for rotations about the origin would be

How is a dilation different from all the other transformations we've looked at? It is not an isometry (congruence mapping). In other words, the pre-image and the image are different sizes. Dilations do not preserve size, only shape.

As participants work on question 9, they may ask “If I rotate by θ and then by α shouldn't this be equivalent to a rotation by $\theta + \alpha$? So, can we reconcile this with the general rotation matrix?”

That is, it should be true that $R_{\theta+\alpha} = R_\alpha \cdot R_\theta$. This allows some foreshadowing to trigonometric identities involving the sine and cosine of sums of angles.

$$\begin{aligned} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} &= \begin{bmatrix} \cos \alpha \cos \theta - \sin \alpha \sin \theta & -\cos \alpha \sin \theta - \sin \alpha \cos \theta \\ \sin \alpha \cos \theta + \cos \alpha \sin \theta & \cos \alpha \cos \theta - \sin \alpha \sin \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos (\theta + \alpha) & -\sin (\theta + \alpha) \\ \sin (\theta + \alpha) & \cos (\theta + \alpha) \end{bmatrix} \end{aligned}$$

Ask participants to discuss their solutions from the Reflect and Apply Task and conduct a class discussion of any discrepancies that arise in the group discussions.

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2. Enter matrix A into your graphing calculator. Then, enter matrix B, below into your calculator.

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3. Find the matrix product, BxA. The result will be another set of ordered pairs. On the same graph paper used in question 1, sketch the polygon formed by connecting the coordinates in matrix BxA. How are the two polygons related?

4. Test your hypothesis by creating several other polygon matrices at your table and finding the product of your new matrices with matrix B. Then plot the results. Record your matrices and the resulting products below. Use the MATPLOT graphing calculator program to help expedite this process. To use this program, matrix A is always the ordered pair matrix and matrix B is the transforming matrix.

5. Algebraically, how do we define a reflection across the y-axis?

$$(x, y) \rightarrow (\underline{\hspace{2cm}}, \underline{\hspace{2cm}})$$

6. Find the product of the two matrices below. How does this product relate to the definition in question 5? How does this relate to your hypothesis from question 4?

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} =$$

7. Can you change matrix B so that, when multiplied by a matrix A, the polygon will be reflected across the x-axis? What is the new matrix B?
8. How is a rotation about the origin of 180° related to reflections across the x- and y-axes? Can you use this information to find a matrix, RO, 180° , that would rotate an object around the origin by 180° ?
9. Find transformation matrices for the following:
RO, 90°
RO, 270°
RO, 360°
RO, θ
Dilation, DO, 3

Reflect and Apply

1. How many points determine a line?
2. Find the equation of the line that passes through the points $(0, 0)$ and $(1, 12)$. Graph the line on your graphing calculator.
3. How many non-collinear points are required to determine a quadratic equation?
4. Find the quadratic equation that passes through the points $(0, 0)$, $(1, 12)$ and $(3, 6)$. (Set up a system of 3 equations and 3 unknowns. Then use matrices to solve for the unknowns.) Add this graph to the graph above.
5. How many non-collinear points are required to determine a cubic equation?
6. Find the cubic equation that passes through the points $(0, 0)$, $(1, 12)$, $(3, 6)$ and $(-1, -14)$. Add this graph to the graph above.