TASK 2.3.2: DOES IT MATTER WHERE THE DAY BEGINS?

Solutions

Consider the situation explored in Tasks 2.3.1a and 2.3.1b.

1. Suppose that you know the initial distribution of the drivers. Say one-third of the drivers begin the day at each location. Write a matrix that represents the driver distribution.

   $$D_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

2. How could this matrix be used to find the distribution of drivers after one delivery?
   $T \cdot D_0 = D_1$

   $$\begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} .400 \\ .267 \\ .333 \end{bmatrix}$$

   The answer matrix represents the distribution of the drivers after the first delivery.

3. What is the distribution after 2 deliveries? $T \cdot D_1 = D_2$

4. How will the drivers be distributed at noon if their shift starts at 8:00? Assume that they all made deliveries from the beginning of the work day and they did not take any breaks. Note: One shift lasts 30 minutes.
   There are eight shifts from 8:00 to 12:00.

   $T \cdot D_0 = D_1$
   $T \cdot D_1 = D_2$
   $T \cdot D_2 = D_3$
   But $T \cdot T \cdot D_0 = D_2$ and $T \cdot T \cdot T \cdot D_0 = D_3$, so $T^n \cdot D_0 = D_n$

   $$\begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix}^8 \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} .38888875 \\ .27777792 \\ .33333333 \end{bmatrix}$$

5. How does the nth distribution, D_n, relate to the transition matrix after n deliveries?

$$T^n = \begin{bmatrix} .38888888 & .38888888 & .38888888 \\ .27777777 & .27777777 & .27777777 \\ .33333333 & .33333333 & .33333333 \end{bmatrix}$$

$$D_n = \begin{bmatrix} .38888888 \\ .27777777 \\ .33333333 \end{bmatrix}$$

Since $T^n \cdot D_0 = D_n$, *we have* $\begin{bmatrix} .38888888 & .38888888 & .38888888 \\ .27777777 & .27777777 & .27777777 \\ .33333333 & .33333333 & .33333333 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

Multiplying row 1 by the distribution matrix, we get $.3888 \cdot \frac{1}{3} + .3888 \cdot \frac{1}{3} + .3888 \cdot \frac{1}{3} = .3888\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) = .3888(1) = .3888$. *Similar calculations hold true for the other two rows. As long as the sum of the entries of the columns in the distribution matrix totals one, then the nth distribution will approach entries for any column of the nth transition matrix.*

6. Find the long-term distribution for the initial distributions given to your group. Describe what the initial distribution for your group means in terms of the problem. Make up a distribution of your own and try it.
There are several listed to try. Participants must choose a number for the exponent for the probability matrix. For these solutions, n = 10.

A. $\begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix}^{10} \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} = \begin{bmatrix} .3888888974 \\ .2777777693 \\ .3333333333 \end{bmatrix}$

Half of the drivers start at the airport, one-quarter of the drivers start at the Arlington location and one-quarter of the drivers start at the Grand Prairie location.

B. $\begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix}^{10} \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix} = \begin{bmatrix} .3888888548 \\ .2777778119 \\ .3333333334 \end{bmatrix}$

Half of the drivers start at the Arlington location and half start at the Grand Prairie location.

C. $\begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix}^{10} \begin{bmatrix} .7 \\ .2 \\ .1 \end{bmatrix} = \begin{bmatrix} .3888889128 \\ .2777777539 \\ .3333333333 \end{bmatrix}$

Seventy percent of the drivers start at the airport, twenty percent start at the Arlington location, and ten percent start at the Grand Prairie location.

D. $$\begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix}^{10} \begin{bmatrix} .4 \\ .3 \\ .3 \end{bmatrix} = \begin{bmatrix} .38888888889 \\ .27777777778 \\ .33333333333 \end{bmatrix}$$

Forty percent of the drivers start at the airport location and thirty percent start at each of the other two locations.

E. $$\begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix}^{10} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .3888889401 \\ .2777777266 \\ .3333333333 \end{bmatrix}$$

All of the drivers start at the airport.

It does not matter where the drivers begin the day, it will not take many deliveries for the drivers to reach the equilibrium state.

7. How does the initial distribution affect the long-term trend?
   The long term trend is the same, regardless of the initial distribution.

Math notes
This continuation of Tasks 2.3.1a and 2.3.1b provides interesting links between the matrix iterations and the fact that the entries in the matrix are probabilities.

Teaching notes
It may be helpful to color-code the numbers on the diagram to each starting location. For example, choose red for DFW Airport. Then the 30% leading of and back into the airports is circled in red as are the 30% leading to Grand Prairie and 40% leading to Arlington.

Teacher Task 2.3.2 looks at different initial distributions. Participants should see fairly quickly that each new distribution vector approaches the steady state fairly quickly.

Some participants may prefer to try working the initial distribution of thirds by using an actual number of drivers. For example, suppose the car rental company employs 60 drivers. Then 20 will be at each location at the beginning. Let them try the problem with a number. Have them go back and try the problem with the given distribution. Discuss how $\left(\frac{1}{3} \;\; \frac{1}{3} \;\; \frac{1}{3} \right)$ is a generalized representation that applies to any number of drivers.

In question 6, assign each group one of the distributions included on the transparency. A general class discussion should follow question 7 to discuss the fact that in the long-run the initial distribution of drivers does not affect the distribution.

Distributions for Teacher Task 2.3.2, question 6.

A. [½ ¼ ¼]

B. [0 ½ ½]

C. [.7 .2 .1]

D. [.4 .3 .3]

E. [1 0 0]

TASK 2.3.2: DOES IT MATTER WHERE THE DAY BEGINS?

Consider the situation explored in Tasks 2.3.1a and 2.3.1b.

1. Now, suppose that you know the initial distribution of the drivers. Say one-third of the drivers begin the day at each location. Write a matrix that represents the driver distribution.

2. How could this matrix be used to find the distribution of drivers after one delivery?

3. What is the distribution after two deliveries?

4. How will the drivers be distributed at noon if their shift starts at 8:00? Assume that they all made deliveries from the beginning of the work day and they did not take any breaks. Note: One shift lasts 30 minutes.

5. How does the nth distribution, D_n, relate to the probability matrix after n deliveries?

6. Find the long-term distribution for the initial distributions given to your group. Describe what the initial distribution for your group means in terms of the problem. Make up a distribution of your own and try it.

7. How does the initial distribution affect the long-term trend? Explain.