TASK 2.3.1A: CAR RENTAL MODEL: AN APPLICATION OF MATRICES

Solutions

A certain car rental agency has three locations in the Arlington-Grand Prairie area. They are labeled on the diagram as:

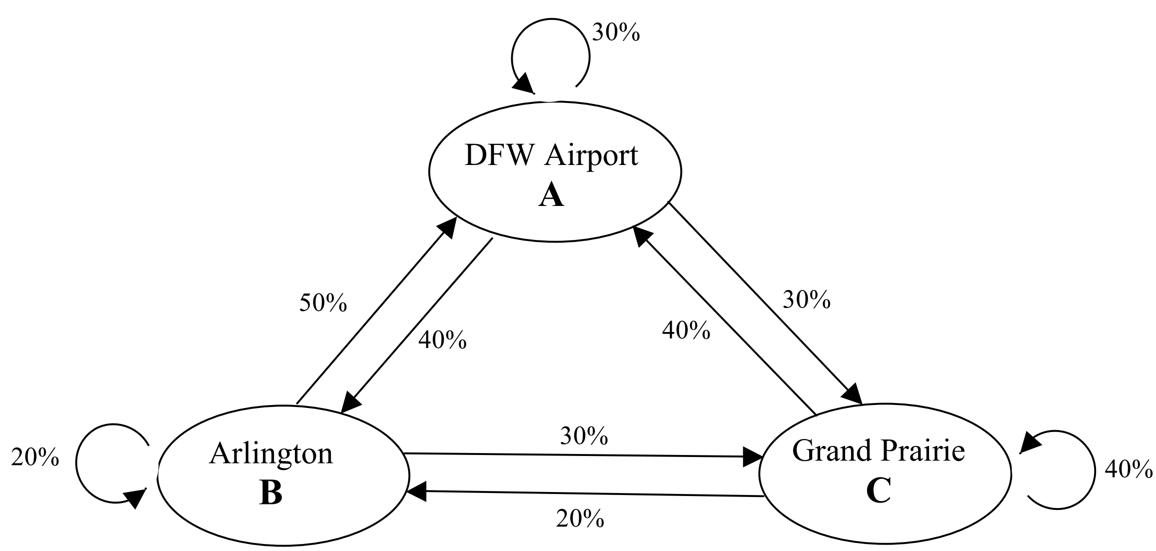
(A) DFW Airport
(B) downtown Arlington
(C) downtown Grand Prairie.

One group of drivers makes deliveries for all three locations. A driver delivers the rental car to a customer. The customer then returns the driver to the nearest agency location. The location of a specific driver is determined only by the driver's *last location*.

The average length of time it takes for a driver to make a delivery and get to their next location is 30 minutes. The agency has determined where the deliveries are made from each location.

- Of the calls sent to the DFW airport location, 30% are delivered in the DFW area, 40% are delivered in the Arlington area, and 30% are delivered in the Grand Prairie area.
- Of the calls sent to the Arlington location, 50% are delivered in the DFW area, 20% are delivered in the Arlington area, 30% are delivered in the Grand Prairie location.
- Of the calls sent to the Grand Prairie location, 40% are delivered in the DFW area, 20% are delivered in the Arlington area, and 40% are delivered in the Grand Prairie area.

The diagram below illustrates these findings.



1. What is the probability (P) that a driver returns to the DFW Airport (location A) after making one delivery? If we think of this as a two-step process, we can write the probability symbolically. Let P("Starting location" "Finish location") represent the probability that a driver will start at particular location and finish at a particular location. For example, P(XY) means that a driver starts at location X and ends at location Y. Write an expression such as P(XY) for this driver that begins at the DFW location and finishes there after one delivery before calculating the probability. *P(AA) = .3*

2. Write, in words, what P(AB) and P(AC) represent and then calculate the probabilities. *P(AB) represents the probability that a driver who starts at DFW will be at the Arlington location after one delivery. P(AB) = .4, P(AC) = .3*

3. Determine (list) the probabilities for the remaining one-trip START FINISH combinations.
   P(BA) = .5 *P(CA) = .4*
   P(BB) = .2 *P(CB) = .2*
   P(BC) = .3 *P(CC) = .4*

4. Write, in words, what $P(AC \cup AA)$ represents and calculate the probability. This question really means "what is the probability that a driver who starts at location A will be at location B *or* location C after one delivery?". $P(AC \cup AA)$ *represents the probability that a driver who starts at location A will be at location C or A after one delivery.*
   P(AB ⋃ AC) = P(AB) + P(AC) *Probability of mutually exclusive events*
   = (.4) + (.3)
   = .7

5. Write, in words, what $P(BA \cap AC)$ represents and calculate the probability. This question really means "what is the probability that a driver who starts at location A will be at location B after the first delivery *and* at location C after the second delivery?".
   $P(BA \cap AC)$ *represents the probability that a driver starts at location B and ends up at location A after the first delivery and location C after the second delivery.*

   P(AB ⋂ BC) = P(AB)P(BC) *Probability of independent events*

TASK 1B: WHERE IS THE DRIVER GOING?

Solutions

1. What is the probability that a driver who begins at location A will be back at location A after exactly two deliveries? Write the solution symbolically before you calculate the answer.
   P(AA) P(AA) + P(AB) P(BA) + P(AC) P(CA) =
   (.3)(.3) + (.4)(.5) + (.3)(.4) =
   .09 + .2 + .12 = .41
2. What is the probability that a driver who begins at location A will be at location B after exactly two deliveries?
   P(AA) P(AB) + P(AB) P(BB) + P(AC) P(CB) =
   (.3)(.4) + (.4)(.2) + (.3)(.2) =
   .12 + .08 + .06m = .26
3. Look back at your calculations and symbolic setup in 1 and 2. Try to find a pattern that would allow you to set up matrices (row and column) to solve problems of this type. Describe your method. Now, set up the matrix notation for the probability of beginning at A and ending at A after exactly two deliveries. Then do the same for starting at A and ending a B after exactly two deliveries. Show that this agrees with your solutions in 1 and 2.
   A 1x3 matrix with the destination probabilities and a 3x1 matrix with the beginning probabilities. While the matrices could be set up another way, this form works out nicely in further calculations.

   $$A \text{ to } A \begin{bmatrix} .3 & .5 & .4 \end{bmatrix} \begin{bmatrix} .3 \\ .4 \\ .3 \end{bmatrix} = \begin{bmatrix} .41 \end{bmatrix} \quad A \text{ to } B \begin{bmatrix} .4 & .2 & .2 \end{bmatrix} \begin{bmatrix} .3 \\ .4 \\ .3 \end{bmatrix} = \begin{bmatrix} .26 \end{bmatrix}$$

   Encourage participants to work this out by hand. We will use the calculator to perform matrix operations when we have set up the 3X3 matrix.

4. To help organize the probabilities for starting at any one of the three destinations and ending at any of the three destinations, we use a transition matrix T. In a table format, we have

	Start at A	*Start at B*	*Start at C*
Finish at A	*.3*	*.5*	*.4*
Finish at B	*.4*	*.2*	*.2*
Finish at C	*.3*	*.3*	*.4*

   The corresponding transition matrix is $\mathbf{T}: \begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix}$ In words, describe what the entries in this matrix represent. *The entry a_{ij} for the jth column and the ith row represents the probability of beginning at the location specified in the jth column and finishing in the location specified by the ith row.*

5. Interpret the following matrix multiplication that uses the transition matrix from problem 4:

   $$\begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix}\begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix} = \begin{bmatrix} .41 & .37 & .38 \\ .26 & .3 & .28 \\ .33 & .33 & .34 \end{bmatrix}.$$

   What do the entries in the resulting matrix represent?

   Row 1 by Column 1

   P(AA) P(AA) + P(AB) P(BA) + P(AC) P(CA) = P(AA) P(AA) + P(BA) P(AB)+ P(CA) P(AC) =

   (.3)(.3) + (.4)(.5) + (.3)(.4) = (.3)(.3) + (.5)(.4) + (.4)(.3) =

   .09 + .2 + .12 = .41

   Symbolically, $T{\cdot}T = T^2$

   The entry a_{11} *in the resulting matrix represents that probability that a driver begins his trip at A and ends his trip after two deliveries at A.*

6. How would you calculate the probability that a driver would be at a certain location after 5 deliveries? After 10 deliveries? Keep as many decimal places as possible to retain accuracy.

   $$T^5 = \begin{bmatrix} .38873 & .38905 & .38894 \\ .27794 & .27762 & .27772 \\ .33333 & .33333 & .33334 \end{bmatrix} \qquad T^{10} = \begin{bmatrix} .38888894 & .38888884 & .38888887 \\ .27777773 & .27777783 & .27777779 \\ .33333333 & .33333333 & .33333333 \end{bmatrix}$$

7. What do you notice about the matrices as more and more deliveries are made?

   The transition probabilities reach a steady state.

8. What is the sum of any column in each of the matrices found in Part 5? How does this number relate to probabilities?

   The columns add to 1. In words, the column represents beginning at a location and finishing at one of the three locations. The probability that this happens is 1 (100%).

9. How could you use the transition matrix to find the probability that a driver would be at a certain location after n deliveries?

   $\mathbf{T^n}$

Math notes
Tasks 2.3.1a and 2.3.1b provide a review of probability embedded in the context of matrix algebra.

This problem is an example of a Markov Chain. The problem setting involves a finite set of states. The probability of movement from the *jth* state to the *ith* state is represented by p_{ij}, where $0 \le p_{ij} \le 1$.

A probability of $p_{ij} = 0$ means that the member (driver) will not change from the *jth* state to the *ith* state. A probability of $p_{ij} = 1$ means that the member (driver) is certainly going to change from the *jth* state to the *ith* state.

Definition: For a Markov chain with *n* states, the state vector is a column vector whose i^{th} component represents the probability that the system is in the i^{th} state at that time. The sum of the state vector is 1. If p_{ij} is the probability of movement (transition) from one state j to state I, the matrix T = [p_{ij}] is called the transition matrix.

$$\overbrace{\mathrm{S}_1 \quad \mathrm{S}_2 \quad \cdots \quad \mathrm{S}_n}^{\text{From}}$$

$$T = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \left.\begin{matrix} \mathrm{S}_1 \\ \mathrm{S}_2 \\ \vdots \\ \mathrm{S}_n \end{matrix}\right\} To$$

For problem six it is important to bring out that not only do the probabilities reach a steady state, but that time there is movement, the next movement depends only on the previous position.

Teaching notes
The goal of these tasks is to provide a glimpse into some of the uses of matrix algebra. In particular, this application related to Markov chains, allows participants to reason through their calculations and setup of the matrices used in modeling for this situation.

Task 2.3.1a sets up the context of the problem. Questions one through four on Task 2.3.1a are a basic probability review. The facilitator may assign the participants to review probability, including the probability of independent events and the probability of mutually excusive events before coming to class. If the participants have a strong grasp of probability, the facilitator may ask the participants to read the problem, then skip to Teacher Task 2.3.1b.
Q What does it mean for two events to be independent? Q
Q What does it mean for two events to be mutually exclusive? Q
Q What operation is needed for mutually exclusive events? Q

As participants work in groups on this task, pass out a transparency to each group for 1-5 in Task 2.3.1a. Assign each group the responsibility of presenting one from 1-5 in Task 2.3.1a. In Task 2.3.1b, ask each group to present their findings to 3 before completing the rest of the task. Some scaffolding before participants can complete 6, may be necessary—ask participants to revisit their verbal explanation from 5 and consider the problem of determining the probability that a driver will end up at a certain location after 3 deliveries.

Task 2.3.1b is where the participants begin the real work of setting up and solving the problem. Before beginning question 3 in Task 1b, discuss matrix multiplication. Discussion of their verbal and written explanations are important to prevent this from becoming an exercise in manipulating matrices without understanding the context.

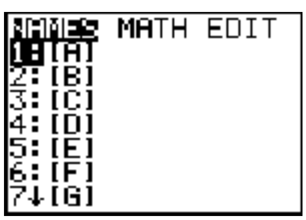
Some participants may want to write the matrix in reverse order with START labeling the rows and FINISH labeling the columns. There are various reasons why this is not as convenient mathematically. You may want to explain that the mathematics will be easier later in the problem if the START labels the columns and FINISH labels the rows. Another approach would be to explain that by convention, probabilities are set up in this way. However, the problem will work out either way. The matrix multiplication will just occur from different sides.

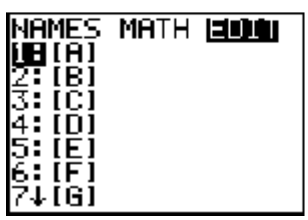
Until question 5 in Task 1b, participants should be encouraged to write their calculations (both symbolic and numeric) by hand.

Technology notes
Before beginning this activity it is helpful for the instructor to review for him/herself the procedure for entering matrices into the calculator.

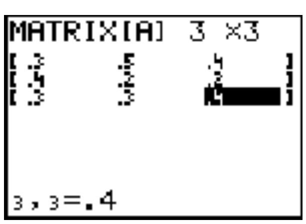
Brief Notes
All matrix arithmetic can be performed on a graphing calculator.
2nd MATRIX, EDIT, 3X3
Enter the state values.
2nd QUIT (Home Screen)
2nd MATRIX, 1:[A] ^ (exponent values)

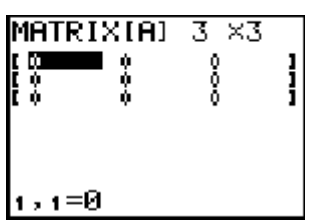
Full Notes
To enter a matrix on the TI-83 plus, press 2nd x^{-1}. Next, right arrow over to EDIT and press ENTER.



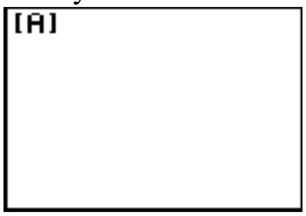


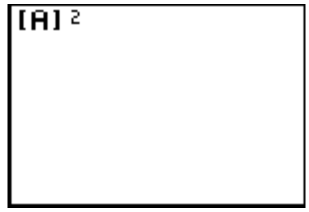
Set the matrix size to 3X3. Enter the matrix values.

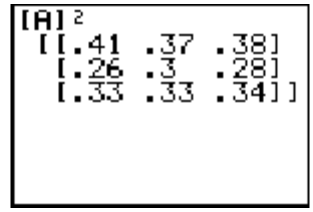




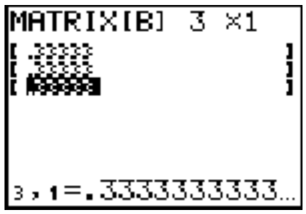
To calculate T^2 press 2^{nd} MODE to go back to home screen. Press 2^{nd} x^{-1} ENTER. Press the x^2 key. ENTER.

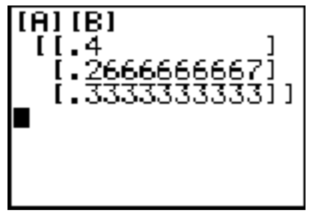






To multiply a 3X3 matrix by 3X1 matrix, set up matrix B.





TASK 2.3.1A: CAR RENTAL MODEL: AN APPLICATION OF MATRICES

A certain car rental agency has three locations in the Arlington-Grand Prairie area. They are labeled on the diagram as:

(A) DFW Airport
(B) downtown Arlington
(C) downtown Grand Prairie.

One group of drivers makes deliveries for all three locations. A driver delivers the rental car to a customer. The customer then returns the driver to the nearest agency location. The location of a specific driver is determined only by the driver's *last location*.

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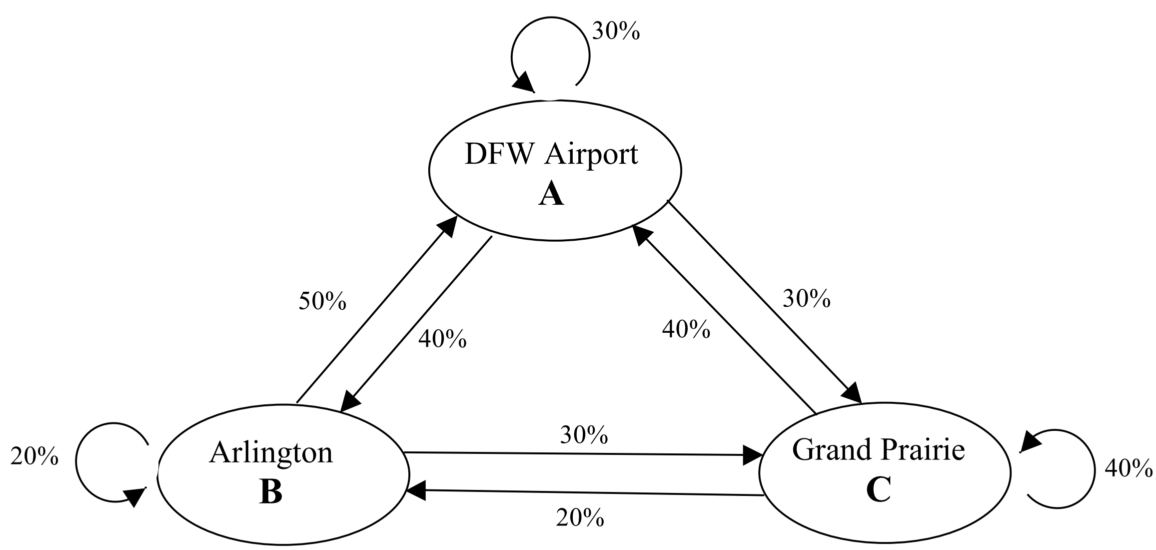
The diagram below illustrates these findings.



1. What is the probability (P) that a driver returns to the DFW Airport (location A) after making one delivery? If we think of this as a two-step process, we can write the probability symbolically. Let P("Starting location" "Finish location") represent the probability that a driver will start at particular location and finish at a particular location. For example, P(XY) means that a driver starts at location X and ends at location Y. Write an expression such as P(XY) for this driver that begins at the DFW location and finishes there after one delivery before calculating the probability.

2. Write, in words, what P(AB) and P(AC) represent and then calculate the probabilities.

3. Determine (list) the probabilities for the remaining one-trip START FINISH combinations

4. Write, in words, what $P(AC \cup AA)$ represents and calculate the probability.

5. Write, in words, what $P(BA \cap AC)$ represents and calculate the probability.

TASK 2.3.1B: WHERE IS THE DRIVER GOING?

1. What is the probability that a driver who begins at location A will be back at location A after exactly two deliveries? Write the solution symbolically before you calculate the answer.

2. What is the probability that a driver who begins at location A will be at location B after exactly two deliveries?

3. Look back at your calculations and symbolic set up in 1 and 2. Try to find a pattern that would allow you to set up matrices (row and column) to solve problems of this type. Describe your method. Now, set up the matrix notation for the probability of beginning at A and ending at A after exactly two deliveries. Then do the same for starting at A and ending a B after exactly two deliveries. Show that this agrees with your solutions in 1 and 2.

4. To help organize the probabilities for starting at any one of the three destinations and ending at any of the three destinations we use a transition matrix T. In a table format, we have

	Start at A	Start at B	Start at C
Finish at A	.3	.5	.4
Finish at B	.4	.2	.2
Finish at C	.3	.3	.4

The corresponding transition matrix is $\mathbf{T}: \begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix}$ In words, describe what the entries in this matrix represent.

5. Interpret the following matrix multiplication that uses the transition matrix from problem 4:

$$\begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix} \begin{bmatrix} .3 & .5 & .4 \\ .4 & .2 & .2 \\ .3 & .3 & .4 \end{bmatrix} = \begin{bmatrix} .41 & .37 & .38 \\ .26 & .3 & .28 \\ .33 & .33 & .34 \end{bmatrix}.$$

What do the entries in the resulting matrix represent?

6. How would you calculate the probability that a driver would be at a certain location after 5 deliveries? After 10 deliveries? Keep as many decimal places as possible to retain accuracy.

7. What do you notice about the matrices as more and more deliveries are made?

8. What is the sum of any column in the matrix? How does this number relate to the fact that they are probabilities?

9. How could you use the transition matrix to find the probability that a driver would be at a certain location after *n* deliveries?