Teaching Phases and Standard Algorithms in the Common Core State Standards or From Strategies to Variations in Writing the Standard Algorithms

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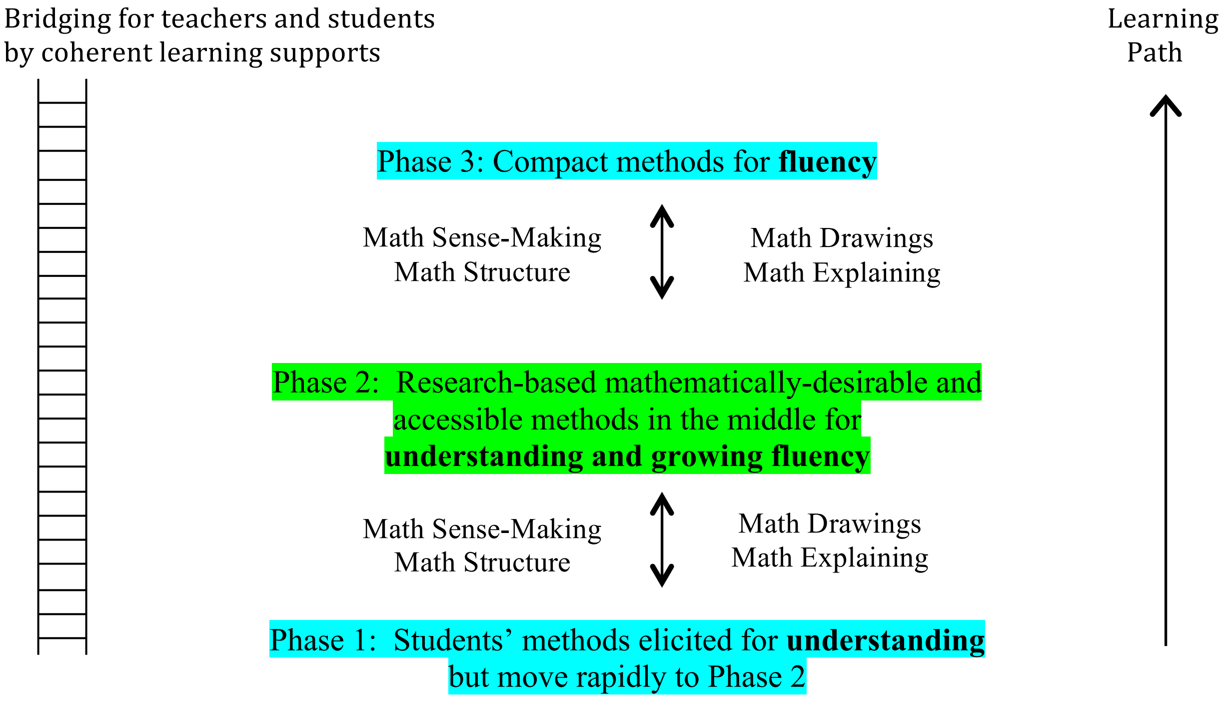
This talk is based on

the CCSS-M standards,

***The NBT Progression for the Common Core State Standards* by The Common Core Writing Team (7 April, 2011), commoncoretools.wordpress.com, and**

Fuson, K. C. & Beckmann, S. (Fall/Winter, 2012-2013). Standard algorithms in the Common Core State Standards. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, 14 (2)*, 14-30 (which is posted at http://www.math.uga.edu/~sybilla/ as is this talk file).

Learning Path Teaching-Learning:
Differentiating within Whole-Class Instruction by Using the Math Talk Community



Bridging for teachers and students
by coherent learning supports

Learning
Path

Phase 3: Compact methods for **fluency**

Math Sense-Making
Math Structure

Math Drawings
Math Explaining

Phase 2: Research-based mathematically-desirable and accessible methods in the middle for **understanding and growing fluency**

Math Sense-Making
Math Structure

Math Drawings
Math Explaining

Phase 1: Students' methods elicited for **understanding** but move rapidly to Phase 2

Common Core Mathematical Practices

Math Sense-Making about Math Structure using Math Drawings to support Math Explaining

Math Sense-Making: Making sense and using appropriate precision

1 Make sense of problems and persevere in solving them.
6 Attend to precision.

Math Structure: Seeing structure and generalizing

7 Look for and make use of structure.
8 Look for and express regularity in repeated reasoning.

Math Drawings: Modeling and using tools

4 Model with mathematics.
5 Use appropriate tools strategically.

Math Explaining: Reasoning and explaining

2 Reason abstractly and quantitatively.
3 Construct viable arguments and critique the reasoning of others.

The top is an extension of Fuson, K. C. & Murata, A. (2007). Integrating NRC principles and the NCTM Process Standards to form a Class Learning Path Model that individualizes within whole-class activities. National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, 10 (1), 72-91. It is a summary of several National Research Council Reports.

The meaningful development of standard algorithms in the CCSS-M

The CCSS-M conceptual approach to computation is deeply mathematical and enables students to make sense of and use the base ten system and properties of operations powerfully. The CCSS-M focus on understanding and explaining such calculations, with the support of visual models, enables students to see mathematical structure as accessible, important, interesting, and useful.

The relationships across operations are also a critically important mathematical idea. How the regularity of the mathematical structure in the base ten system can be used for so many different kinds of calculation is an important feature of what we want students to appreciate in the elementary grades.

It is crucial to use the Standards of Mathematical Practice throughout the development of computational methods.

Misconceptions about the CCSS-M and the NBT Progression. These are all wrong.

The standard algorithm is the method I learned.

The standard algorithm is the method commonly taught now (the current common method).

There is only one way to write the algorithm for each operation.

The standard algorithm means teaching rotely without understanding.

Teachers or programs may not teach the standard algorithm until the grade at which fluency is specified in the CCSS-M.

Initially teachers or programs may only use methods that children invent.

Teachers or programs must emphasize special strategies useful only for certain numbers.

General methods that will generalize to and become standard algorithms can and should be developed, discussed, and explained initially using a visual model.

The critical area for the initial grade in which a type of multidigit computation is introduced specifies that:

Students develop, discuss, and use efficient, accurate, and generalizable methods to [+ - x ÷].

The standard for the initial grade in which a type of multidigit computation is introduced specifies that students are initially to:

use concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. [Grade 1 addition and Grade 2 addition and subtraction]

or

Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. [Grade 4 multiplication and division]

In the past, there has been an unfortunate and harmful dichotomy suggesting that

***strategy* implies understanding and *algorithm* implies no visual models, no explaining, and no understanding.**

In the past, teaching *the standard algorithm* has too often meant teaching numerical steps rotely and having students memorize the steps rather than understand and explain them.

The CCSS-M clearly do not mean for this to happen, and the NBT Progression document clarifies this by showing visual models and explanations for various minor variations in the written methods for the standard algorithms for all operations.

The word "strategy" emphasizes that computation is being approached thoughtfully with an emphasis on student sense-making. *Computation strategy* as defined in the Glossary for the CCSS-M includes special strategies chosen for specific problems, so a strategy does not have to generalize. But the emphasis at every grade level within all of the computation standards is on efficient and generalizable methods, as in the Critical Areas.

The NBT Progression document summarizes that

the standard algorithm for an operation implements the following mathematical approach
with minor variations in how the algorithm is written:

> decompose numbers into base-ten units and then carrying out single-digit computations with those units using the place values to direct the place value of the resulting number; and
>
> use the one-to-ten uniformity of the base ten structure of the number system to generalize to large whole numbers and to decimals.

To implement a standard algorithm one uses a systematic *written method* for recording the steps of the algorithm.

There are variations in these written methods
  within a country
  across countries
  at different times.

Criteria for Emphasized Written Methods That Should be Introduced in the Classroom

Variations that support and use place value correctly

Variations that make single-digit computations easier, given the centrality of single-digit computations in algorithms

Variations in which all of one kind of step is done first and then the other kind of step is done rather than alternating, because variations in which the kinds of steps alternate can introduce errors and be more difficult.

Variations that keep the initial multidigit numbers unchanged because they are conceptually clearer

Variations that can be done left to right are helpful to many students because many students prefer to calculate from left to right.

The Learning Path Using Helping Step Variations

Some variations of a written method include steps or math drawings that help students make sense of and keep track of the underlying reasoning and are an easier place to start. These variations are important initially for understanding.

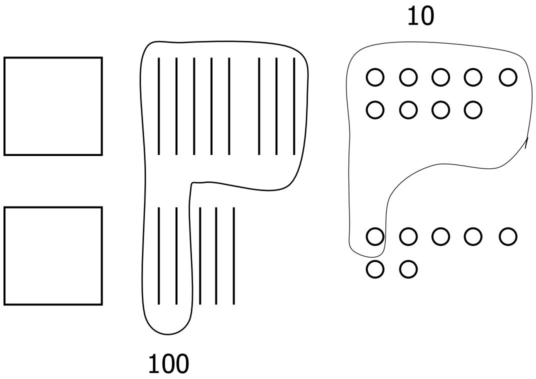
But over time, these longer written methods can be abbreviated into shorter written methods that are variations of writing the standard algorithm for an operation.

This learning path allows students to achieve fluency with the standard algorithm while still being able to understand and explain the shortened method.

Emphasized methods on which students spend significant time must have a clear learning path to some written variation of the standard algorithm. And students who can move to this written variation should be able to do so.

Written Variations of Standard Algorithms

Quantity Model ↔ **Good Variations** **Current Common**



New Groups Below

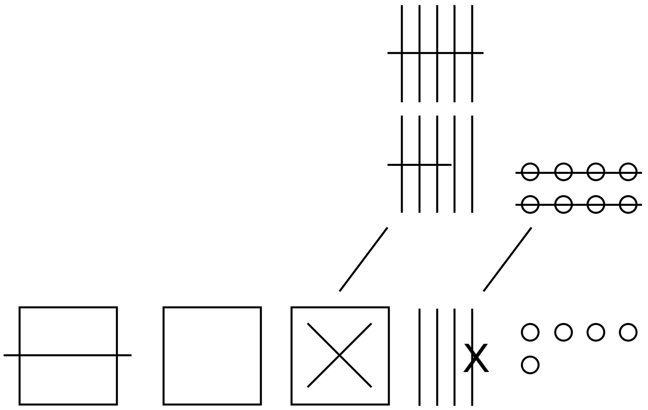
$$\begin{array}{r} 1\ 8\ 9 \\ +\ 1\ 5\ 7 \\ \hline \overset{1}{3}\ \overset{1}{4}\ 6 \end{array}$$

Show All Totals

$$\begin{array}{r} 1\ 8\ 9 \\ +\ 1\ 5\ 7 \\ \hline 2\ 0\ 0 \\ 1\ 3\ 0 \\ 1\ 6 \\ \hline 3\ 4\ 6 \end{array}$$

New Groups Above

$$\begin{array}{r} \overset{1}{1}\ \overset{1}{8}\ 9 \\ +\ 1\ 5\ 7 \\ \hline 3\ 4\ 6 \end{array}$$



Ungroup Everywhere First, Then Subtract Everywhere

Left → Right

$$\begin{array}{r} 13\ \ \ \ \ \\ 2\ \ \cancel{14}\ \ 16 \\ \cancel{3}\ \ \cancel{4}\ \ \cancel{6} \\ -\ 1\ \ 8\ \ 9 \\ \hline 1\ \ 5\ \ 7 \end{array}$$

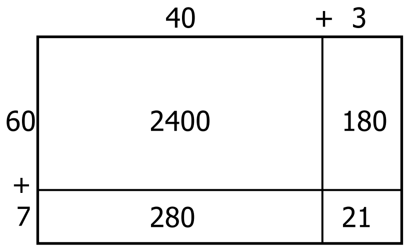
Right → Left

$$\begin{array}{r} 13\ \ \ \ \ \\ 2\ \ \cancel{3}\ \ 16 \\ \cancel{3}\ \ \cancel{4}\ \ \cancel{6} \\ -\ 1\ \ 8\ \ 9 \\ \hline 1\ \ 5\ \ 7 \end{array}$$

R → L Ungroup, Then Subtract, Ungroup, Then Subtract

$$\begin{array}{r} 13\ \ \ \ \ \\ 2\ \ \cancel{3}\ \ 16 \\ \cancel{3}\ \ \cancel{4}\ \ \cancel{6} \\ -\ 1\ \ 8\ \ 9 \\ \hline 1\ \ 5\ \ 7 \end{array}$$

Area Model



Place Value Sections

$$\begin{array}{r} 2\,4\,0\,0 \\ 1\,8\,0 \\ 2\,8\,0 \\ +\ \ 2\,1 \\ \hline \overset{1}{2}\,8\,8\,1 \end{array}$$

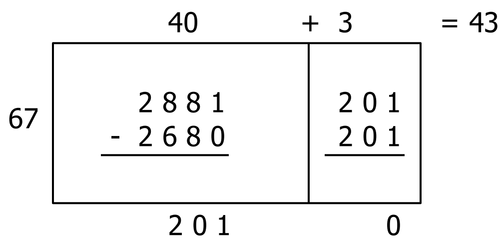
Expanded Notation

$$\begin{array}{rcr} 43 & = & 40 + 3 \\ \times\ \ 67 & = & 60 + 7 \\ \hline 60 \times 40 & = & 2\,4\,0\,0 \\ 60 \times\ \ 3 & = & 1\,8\,0 \\ 7 \times 40 & = & 2\,8\,0 \\ 7 \times\ \ 3 & = & 2\,1 \\ \hline & & \overset{1}{2}\,8\,8\,1 \end{array}$$

1-Row

$$\begin{array}{r} {}^{1}_{2}\ \ \ \\ 4\,3 \\ \times\ \ 6\,7 \\ \hline 3\,0\,1 \\ 2\,5\,8\ \ \\ \hline 2\,8\,8\,1 \end{array}$$

Rectangle Sections



Expanded Notation

$$\begin{array}{r} \left.\begin{array}{r} 3 \\ 4\,0 \end{array}\right]\ 43 \\ 67\,\overline{)\,2\,8\,8\,1} \\ -\ 2\,6\,8\,0 \\ \hline 2\,0\,1 \\ -\ 2\,0\,1 \\ \hline \end{array}$$

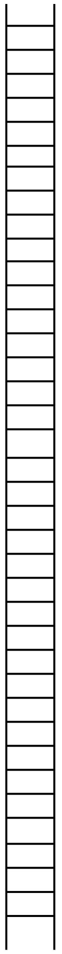
Digit by Digit

$$\begin{array}{r} 4\,3 \\ 67\,\overline{)\,2\,8\,8\,1} \\ -\ 2\,6\,8\ \ \\ \hline 2\,0\,1 \\ -\ 2\,0\,1 \\ \hline \end{array}$$

Learning Path for Multidigit Computation in CCSS-M

Bridging for teachers and students
by coherent learning supports

Learning
Path



Phase 3: Compact methods for **fluency**

Students use any good variation of writing the standard algorithm with no drawing to build fluency. They explain occasionally to retain meanings.

Math Sense-Making
Math Structure

Math Drawings
Math Explaining

Phase 2: Research-based mathematically-desirable and accessible methods in the middle for **understanding and growing fluency**

Students focus on and compare efficient, accurate, and generalizable methods, relating these to visual models and explaining the methods.
They write methods in various ways and discuss the variations.
They may use a helping step method for understanding and/or accuracy.
They choose a method for fluency and begin solving with no drawing.

Math Sense-Making
Math Structure

Math Drawings
Math Explaining

Phase 1: Students' methods elicited for **understanding** but move rapidly to Phase 2

Students develop, discuss, and use efficient, accurate, and generalizable methods and other methods. They use concrete models or drawings that they relate to their written method and explain the reasoning used.

Note. Students may consider problems with special structure (e.g., 98 + 76) and devise quick methods for solving such problems. But the major focus must be on general problems and on generalizable methods that focus on single-digit computations (i.e., that are or will generalize to become a variation of writing the standard algorithm).

What to Emphasize and Where to Intervene as Needed

Grades K to 2 are more ambitious than some/many earlier state standards:

- K: The ten in teen numbers
- G1: + within 100 with composing a new ten; ok if many children still use math drawings; no subtraction without decomposing a ten
- G2: a) +- total ≤100 with composing and decomposing a ten; use math drawings initially, but fluency requires no math drawings
  - b) +- totals 101 to 1,000 with math drawings; vital get mastery by most so that G3 can focus on x÷; intervene with as many as possible to get G2 mastery

Grades 3 to 6 are less ambitious than some/many earlier state standards:

- G3: Fluency for G2 goals +- totals 101 to 1,000, so no math drawings; no new problem sizes so can focus on x÷ [intervene for x÷ all year]
- G4 and G5: a) x only up to 1-digit x 2-, 3-, 4-digits and 2-digits x 2-digits; so not really need mastery of 1-row methods for multiplication [have time for fractions]
- b) division has only the related unknown factor problems; 1-digit divisors G4 and 2-digit divisors G5; fluency G6

The Computation Learning Path

Any method that is taught or used must have a learning path resting on visual models and on explaining the reasoning used. It is not acceptable to teach methods by rote without understanding how place values are used in the methods.

Methods are elicited from students and discussed, but good variations of writing the standard algorithm are introduced early on so that all students can experience them.

Steps in written methods are initially related to steps in visual models.

Experiencing and discussing variations in writing a method is important mathematically.

Students stop making drawings when they do not need them. Fluency means solving without a drawing.

Students drop steps of Helping Step methods when they can move to a short written variation of the standard algorithm for fluency.

FIGURE 1: *Multidigit Addition Methods that Begin with One Undecomposed Number (Count or Add On)*

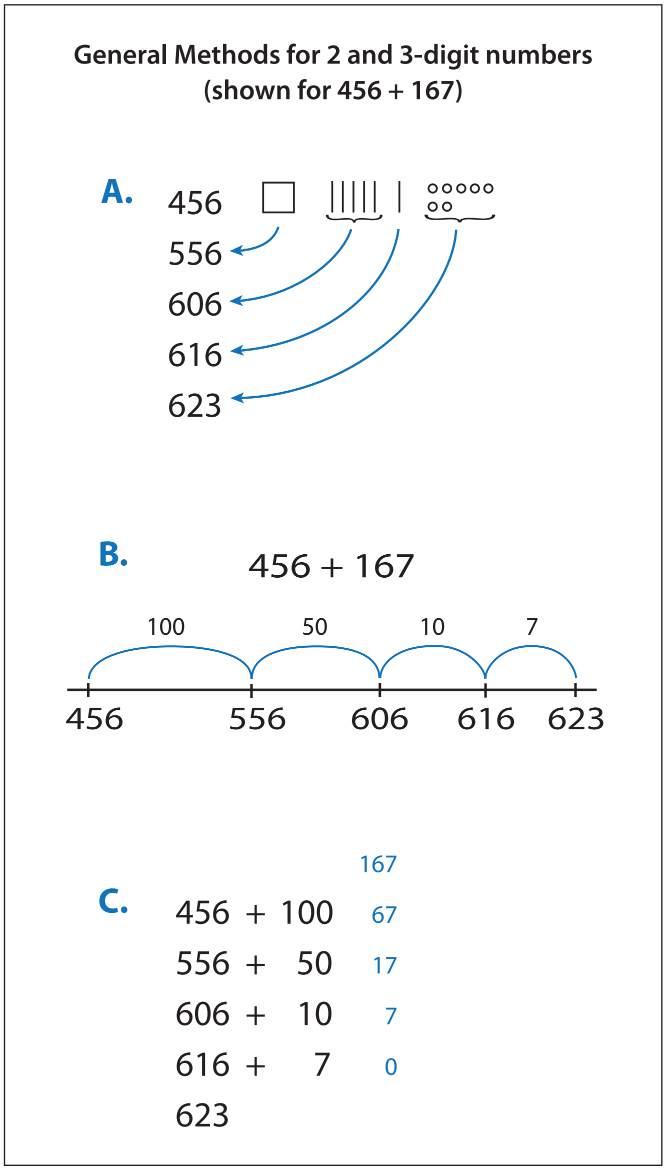
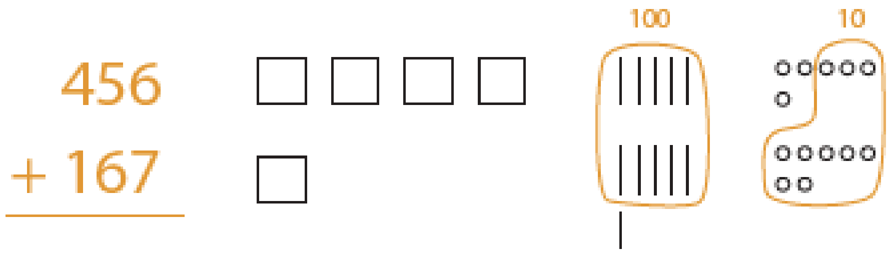


FIGURE 2: *Multidigit Addition Methods that Decompose into Base Ten Units*

Place value drawing for all methods



General methods for 2- and 3-digit numbers

D.
$$\begin{array}{r} 456 \\ +\,167 \\ \hline 500 \\ 110 \\ 13 \\ \hline 623 \end{array}$$

E.
$$\begin{array}{r} 456 \\ +\,167 \\ \hline \overset{1}{6}\overset{1}{2}3 \end{array}$$

F.
$$\begin{array}{r} \overset{5}{\cancel{4}}\overset{6}{\cancel{5}}6 \\ +\,167 \\ \hline 623 \end{array}$$

G.
$$\begin{array}{r} \overset{1}{4}\overset{1}{5}6 \\ +\,167 \\ \hline 623 \end{array}$$

Methods E, F, and G generalized to 6-digit numbers

E.
$$\begin{array}{r} 456{,}789 \\ +\,167{,}189 \\ \hline \overset{1}{6}\overset{1}{2}3{,}\overset{1}{9}\overset{1}{7}8 \end{array}$$

F.
$$\begin{array}{r} \overset{5}{\cancel{4}}\overset{6}{\cancel{5}}6{,}\overset{8}{\cancel{7}}\overset{9}{\cancel{8}}9 \\ +\,167{,}189 \\ \hline 623{,}978 \end{array}$$

G.
$$\begin{array}{r} \overset{1}{4}\overset{1}{5}6{,}\overset{1}{7}\overset{1}{8}9 \\ +\,167{,}189 \\ \hline 623{,}978 \end{array}$$

Note: Methods E, F and G are all variations in the standard algorithm, but Method E is conceptually clearer and easier.

FIGURE 3: *Multidigit Subtraction Methods that Decompose into Base Ten Units*

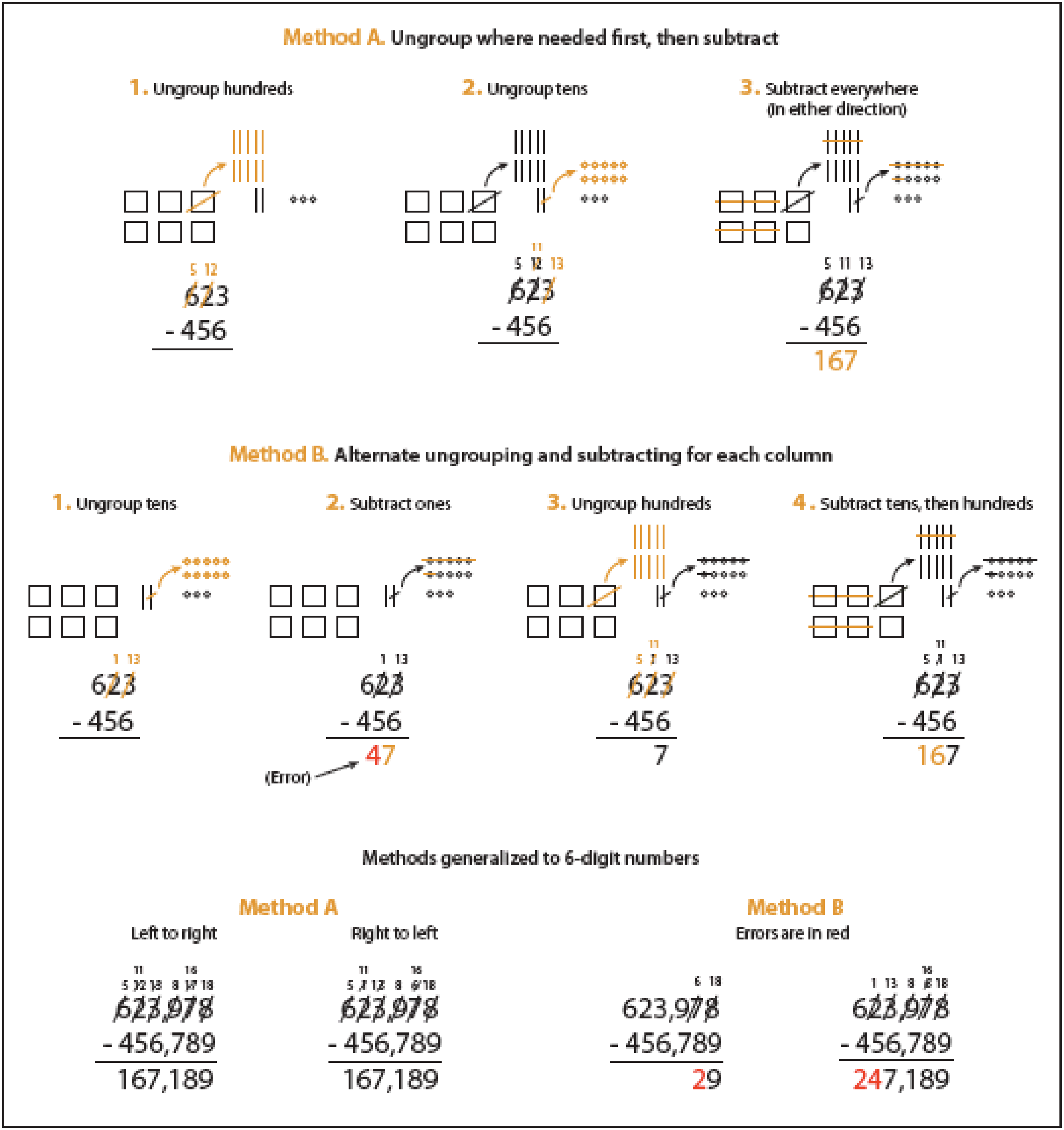
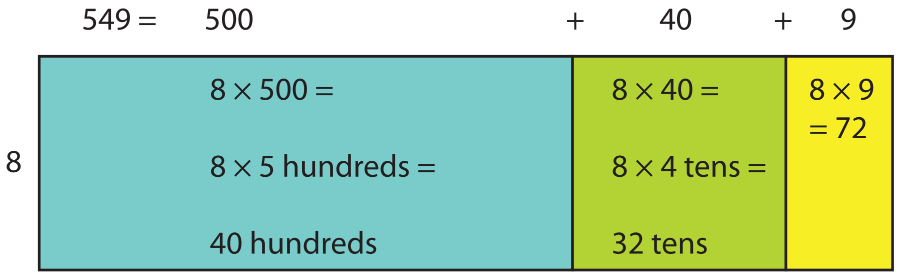


Figure 4: Written Methods for the Standard Multiplication Algorithm, 1-digit × 3-digit

Array/area drawing for 8 × 549



$$\begin{aligned} 8 \times 549 &= 8 \times (500 + 40 + 9) \\ &= 8 \times 500 + 8 \times 40 + 8 \times 9 \end{aligned}$$

Method A:

Left to right
showing the
partial products

$$\begin{array}{rl} 549 & \\ \times\ 8 & \text{thinking:} \\ \hline 4000 & \boxed{8 \times 5 \text{ hundreds}} \\ 320 & \boxed{8 \times 4 \text{ tens}} \\ 72 & \boxed{8 \times 9} \\ \hline 4392 & \end{array}$$

Method B:

Right to left
showing the
partial products

$$\begin{array}{rl} 549 & \\ \times\ 8 & \text{thinking:} \\ \hline 72 & \boxed{8 \times 9} \\ 320 & \boxed{8 \times 4 \text{ tens}} \\ 4000 & \boxed{8 \times 5 \text{ hundreds}} \\ \hline 4392 & \end{array}$$

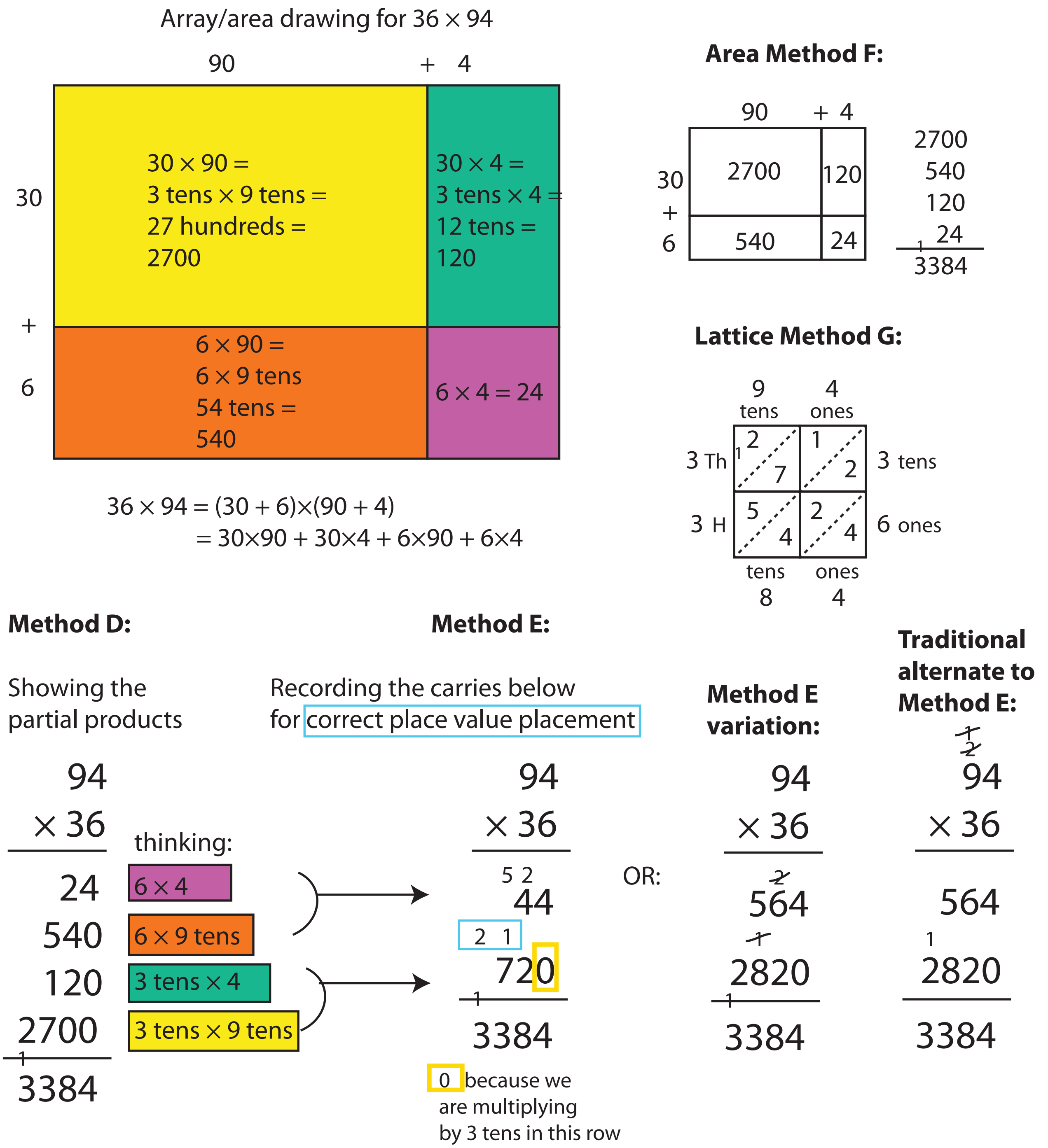
Method C:

Right to left
recording the
carries below

$$\begin{array}{r} 549 \\ \times\ 8 \\ \hline {}^{4\ 3\ 7} \\ 022 \\ \hline 4392 \end{array}$$

Method A proceeds from left to right, and the others from right to left. In Method C, the digits representing new units are written below the line rather than above 549, thus keeping the digits of the products close to each other, e.g., the 7 from 8×9=72 is written diagonally to the left of the 2 rather than above the 4 in 549.

Figure 5: Written Methods for the Standard Multiplication Algorithm, 2-digit × 2-digit



Written Methods D and E are shown from right to left, but could go from left to right.

In Method E, and Method E variation, digits that represent newly composed tens and hundreds in the partial products are written below the line intead of above 94. This way, the 1 from 30 × 4 = 120 is placed correctly in the hundreds place, unlike in the traditional alternate to Method E, where it is placed in the (incorrect) tens place. In the Method E variation, the 2 tens from 6 × 4 = 24 are added to the 4 tens from 6 × 90 = 540 and then crossed out so they will not be added again; the situation is similar for the 1 hundred from 30 × 4 = 120. In Method E, all multiplying is done first and then all adding. In the variations following Method E, multiplying and adding alternate, which is more difficult for some students.

Note that the 0 in the ones place of the second line of Method E is there because the whole line of digits is produced by multiplying by 30 (not 3). This is also true for the following two methods.

Figure 6: Further Written Methods for the Standard Multiplication Algorithm, 2-digit × 2-digit

Method H:

A misleading abbreviated method

$$\begin{array}{rrrr} & & {}^{1} & \\ & & {}^{2} & \\ & & 9 & 4 \\ & \times & 3 & 6 \\ \hline & 5 & 6 & 4 \\ {}^{1}2 & 8 & 2 & \\ \hline 3 & 3 & 8 & 4 \end{array} \quad \xleftarrow{\hspace{2em}}$$

From 30 × 4 = 120.
The 1 is 1 hundred,
not 1 ten.

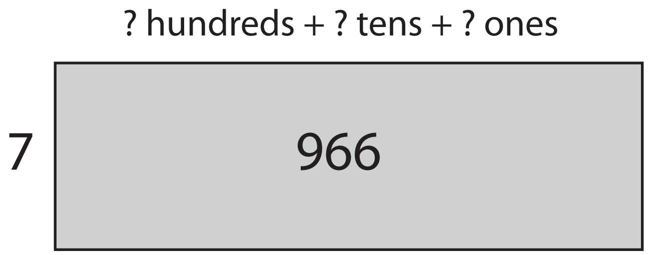
Helping Steps Method I:

$$\begin{array}{rcrcr} 94 & = & 90 & + & 4 \\ \times 36 & = & 30 & + & 6 \\ \hline \end{array}$$

$$\begin{array}{rcr} 30 \times 90 & = & 2700 \\ 30 \times 4 & = & 120 \\ 6 \times 90 & = & 540 \\ 6 \times 4 & = & 24 \\ \hline & & {}^{1} \\ & & 3384 \end{array}$$

Figure 7: Written methods for the standard division algorithm, 1-digit divisor

Area/array drawing for 966 ÷ 7

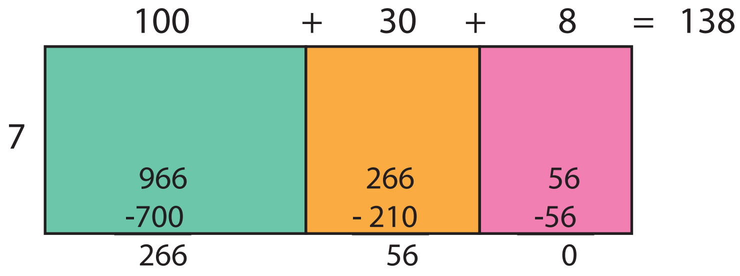


$$\begin{array}{r} ??? \\ 7\overline{)966} \end{array}$$

Thinking: A rectangle has area 966 and one side of length 7. Find the unknown side length. Find hundreds first, then tens, then ones.

$$\begin{aligned} 966 &= 7\times100 + 7\times30 + 7\times8 \\ &= 7\times(100 + 30 + 8) \\ &= 7\times138 \end{aligned}$$

Method A:



$$\begin{array}{r} \left.\begin{array}{r} 8 \\ 30 \\ 100 \end{array}\right] 138 \\ 7\overline{)966} \\ -\,700 \\ \hline 266 \\ -\,210 \\ \hline 56 \\ -\,56 \\ \hline 0 \end{array}$$

Method B:

$$\begin{array}{r} 138 \\ 7\overline{)966} \\ -\,7 \\ \hline 26 \\ -\,21 \\ \hline 56 \\ -\,56 \\ \hline 0 \end{array}$$

Conceptual language for this method:

Find the unknown length of the rectangle;
first find the hundreds, then the tens, then the ones.

The length gets 1 hundred (units); 2 hundreds (square units) remain.

2 hundreds + 6 tens = 26 tens.

The length gets 3 tens (units); 5 tens (square units) remain.

5 tens + 6 ones = 56 ones.

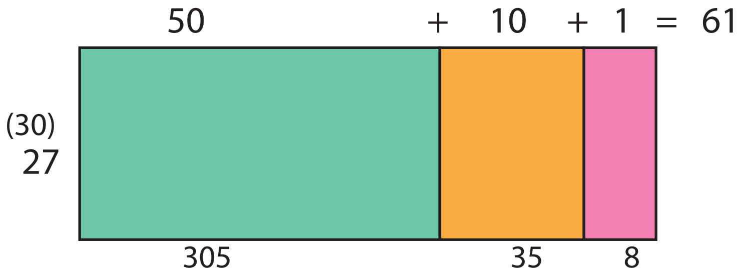
The length gets 8 ones; 0 remains.

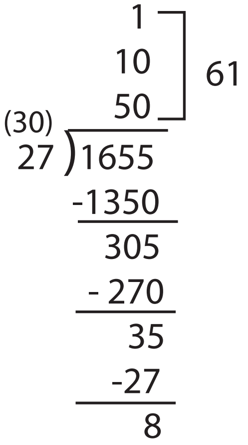
The "bringing down" steps represent unbundling a remaining amount and combining it with the amount at the next lower place.

Figure 8: Written methods for the standard division algorithm,
2-digit divisor

1655÷27

Method A:

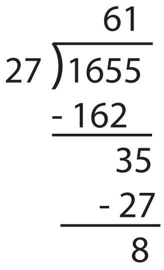




Rounding 27 to 30 produces the underestimate 50 at the first step, but this method allows the division process to be continued.

Method B: Two variations

Erase an underestimate to make it exact.



Change the multiplier but not the underestimated product; then subtract more.