

FUNCTIONS AND EQUATIONS

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Introduction

This note discusses how two kinds of mathematical objects, **functions** and **equations**, are related to each other, and how they are different. This is a relevant subject for those who teach algebra courses and the courses that lead up to algebra, since in recent years there have been significant changes in the way these subjects have been introduced in the curriculum. A generation ago the Algebra I course was primarily a course about equations; functions tended to be viewed as a more advanced subject that appeared in Algebra II and Precalculus. Today the tendency is to introduce functions much earlier, even before Algebra I.

There is much to recommend this early use of functions, provided that it is an appropriately simple and useful concept of function that is developed. A difficulty is that, in this era of transition in the role functions play in the curriculum, there is often considerable inconsistency and vagueness in the way curriculum materials treat functions, equations, and their relationship. This is unfortunate, since understanding the distinct but related roles that functions and equations play can be of considerable help in solving problems.

This note does two things. First, it shows a situation where we start with a function and then proceed to solve an equation based on this function. Second, it reverses the process, showing a situation where we start by solving an equation, and then proceed to generalize the solution by creating a function based on this solution. The examples are purposefully kept very simple so that the interaction between functions and equations can emerge more clearly.

Overview

A **function** expresses a relationship between two quantities, an input x and an output y , such that every input yields a unique output. For example,

$$y = 1.75 \cdot x$$

describes a function that expresses a relationship between y and x . Specifically, the value of y is always 1.75 times the value of x . We **evaluate** a function at a particular input value

of x when we find the value of y that corresponds to this value of x . For example, evaluating this function at the input $x = 9$ gives the output $y = 15.75$.

An **equation** states a condition on a single quantity. For example,

$$1.75 \cdot x = 35$$

is an equation that states a condition on x . Specifically, the condition is that 1.75 times x must equal 35. We **solve** an equation when we find a value (or values) of x that meet the condition. In this case, there is a unique solution to the equation, namely $x = 20$.

Note: The particular function illustrated here is called a **function of one variable**, since there is just one input variable x . For comparison, $z = 1.75 \cdot x + 10 \cdot y$ describes a function of *two* variables, x and y . Similarly, the particular equation illustrated here is called an **equation in one unknown**, where the unknown is the quantity x . For comparison, $35 = 1.75 \cdot x + 10 \cdot y$ states an equation in *two* unknowns, x and y .

In this note we only look at functions of one variable and equations of one unknown.

Starting with a function

Suppose we consider a situation where the cost of gasoline is \$1.75 per gallon. This situation can be represented with a *function*

$$(1) \quad y = 1.75 \cdot x$$

where the input x is the number of gallons, and the output y is the cost in dollars of x gallons. A typical use of such a function would be to evaluate it to find the cost of a certain number of gallons. For example, if we buy 7 gallons of gasoline, we can find the cost in dollars by evaluating this function at $x = 7$:

$$y = 1.75 \cdot (7) = 12.25.$$

We have found the output \$12.25 corresponding to the given input 7 gallons.

Suppose we are interested instead in how much gasoline we can buy for \$35.00. We need to find an input x of the function (1) that corresponds to the given output \$35.00. Putting these values into the function gives this equation in the unknown x :

$$35.00 = 1.75 \cdot x$$

Solving this equation gives the answer, $x = 20$ gallons.

Notice that this equation is based directly on the function (1). The desired output $y = 35.00$ is specified, and the input x is the unknown. In general, asking for the *input* of a function that produces a specified *output* always leads to an equation in exactly this way.

This example was very simple, and the equation was easy to solve. Often, however, equations may be hard to solve even when the corresponding function is easy to evaluate. For example, consider this function:

$$(2) \quad y = x^2 + 1.5 \cdot x$$

We can easily find the *output* this function at any *input* x simply by computing. For example, at the input $x = 7$ the output is

$$(7)^2 + 1.5 \cdot (7) = 59.5$$

However, if we want to find the *input* x that corresponds to the *output* $y = 45$, we need to solve this equation for x :

$$45 = x^2 + 1.5 \cdot x$$

This requires more work than just computing. (In this case, the quadratic formula shows that there are actually two solutions to the equation, $x = 6$ and $x = -7.5$.)

In this section we have shown a situation where we are given a *function*, and have found the input corresponding to a given output of that function by setting up and solving an *equation*. In the next section we show how we might start with an *equation*, and go on to define a *function* based on this equation.

Starting with an equation

Suppose we have to make a trip of 175 miles and we need to get there in 5 hours. How fast do we need to travel? Since the rate of speed r , time t , and distance d satisfy the relationship $d = r \cdot t$, we can represent the problem by letting $d = 175$ miles and $t = 5$ hours, and set up this *equation* in the unknown r .

$$(3) \quad 175 = 5 \cdot r.$$

Solving this equation for the rate of speed r gives $r = 35$ mph.

On the other hand, suppose we have 5 hours in which to make a tour. In general, how fast do we need to travel if we want to cover d miles? Building on the solution

above, we substitute $t = 5$ hours into $d = r \cdot t$ and get the equation $d = 5 \cdot r$. Solving this equation for the rate of speed r gives r as a function of d :

$$(4) \quad r = \frac{d}{5}$$

This shows that the rate of speed r is directly proportional to the distance d .

Notice that this function (4) *generalizes* the solution to the original problem. The original problem asks for the speed needed to go 175 miles in 5 hours. The function (4) tells us how to find the speed required to go *any* number d of miles in 5 hours.

For another variant, suppose that we often have to make a trip of 175 miles. In general, how fast do we need to travel if we want to get there in t hours? We substitute $d = 175$ miles in $d = r \cdot t$ and get the equation

$$175 = r \cdot t.$$

Solving this equation for the rate of speed r gives r as a function of the time t .

$$r = \frac{175}{t}$$

This shows that the rate of speed r is inversely proportional to the time t .

The purpose of the examples in this section has been to illustrate how *equations* can lead to *functions* in the solution of problems. Many problems in algebra can be approached in the same way. We first set up an equation with one unknown r that expresses a relationship among the numbers given in the problem statement. Solving that equation gives a number (or perhaps more than one number) as a solution to the original problem. Then, if we replace one of these numbers with a variable d , and solve the *same equation* for the *same quantity* r in exactly the same way, we get r as a *function of* d . This function represents a generalization of the problem.