

PITFALLS OF OVER-RELIANCE ON CROSS MULTIPLICATION AS A METHOD TO FIND MISSING VALUES

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1. If Kobe bought 17 gallons of gasoline for \$23.63, how much would he have to pay for 11 gallons?

In a traditional approach used for many centuries to solve problems like this, the problem situation is represented as an equality of two ratios of cost to amount:

$$\frac{c_1}{q_1} = \frac{c_2}{q_2}$$

In school mathematics, such an equation is often called a "proportion". Filling in the 3 numerical values gives this equation:

$$\frac{23.63 \text{ dollars}}{17 \text{ gallons}} = \frac{c_2 \text{ dollars}}{11 \text{ gallons}}$$

To solve this equation for c_2 , students are taught to "cross multiply". This means taking each denominator and multiplying it by the numerator of the other side of the equation. (It is equivalent to multiplying each side of the equation by the product of the denominators, and canceling.) Cross-multiplying gives the product $23.63 \cdot 11 = 259.93$. and the equivalent equation:

$$259.93. = c_2 \cdot 17$$

Solving this equation for the cost c_2 gives the correct answer, \$15.29.

A difficulty with using this as the standard or preferred approach to solving such a problem is that the number created by cross multiplying (in this case the number 259.93) has no relevance for the problem situation. Even its units ("dollar gallons") are meaningless.

This would not be so bad, perhaps, except that this cross-multiplying approach *prevents* students from seeing a number that *does* have a great deal of relevance for the problem situation, namely the unit price. To show what we mean, let us illustrate an alternative approach.

In this approach, we divide the cost (\$23.63) by the amount of gasoline it buys (17 gallons) to give the unit price: \$1.39 per gallon. This is an important

quantity in the problem situation. It is the constant value that each ratio above is equal to:

$$\frac{23.63 \text{ dollars}}{17 \text{ gallons}} = \$1.39 \text{ per gallon}$$

To find the unknown value c_2 dollars we multiply the unit price, \$1.39 per gallon, by the given number of gallons, 11, to find the cost of 11 gallons: \$15.29.

$$(1.39 \text{ dollars/gallon}) \cdot (11 \text{ gallons}) = 15.29 \text{ dollars}$$

Notice that computing the unit price gives us an important and useful piece of information about the problem situation in general. If we were asked a variant of the problem, for example, how much would 14 gallons cost?, we could solve it easily: just multiply 14 by the unit price \$1.39 per gallon that we have already found. On the other hand, if we had solved the original problem by cross multiplying, to solve this variant problem we would have to start from scratch (and in the process, we would create another number that has meaningless units and that is useless in any other version of the problem).

Another variant of the original problem might give the same information and ask how many gallons Kobe could buy for some amount of money (say \$20.00.):

2. If Kobe bought 17 gallons of gasoline for \$23.63, how much could he buy for \$20.00?

For this problem, if we knew the "unit amount" (how much gasoline he could buy for 1 dollar), we could easily answer the question. This suggests we set up an equation using the *reciprocals* of the ratios in the equation above.

$$\frac{17 \text{ gallons}}{23.63 \text{ dollars}} = \frac{q_2 \text{ gallons}}{20.00 \text{ dollars}}$$

Here, if we divide 17 gallons by \$23.63 we get the unit amount, 0.719 gallons per dollar. It is the constant value that each ratio above is equal to:

$$\frac{17 \text{ gallons}}{23.63 \text{ dollars}} = 0.719 \text{ gallons per dollar}$$

To find the unknown value q_2 gallons we multiply the unit amount by the given number of dollars, 20, to find the amount that \$20.00 will buy: about 14.38 gallons.

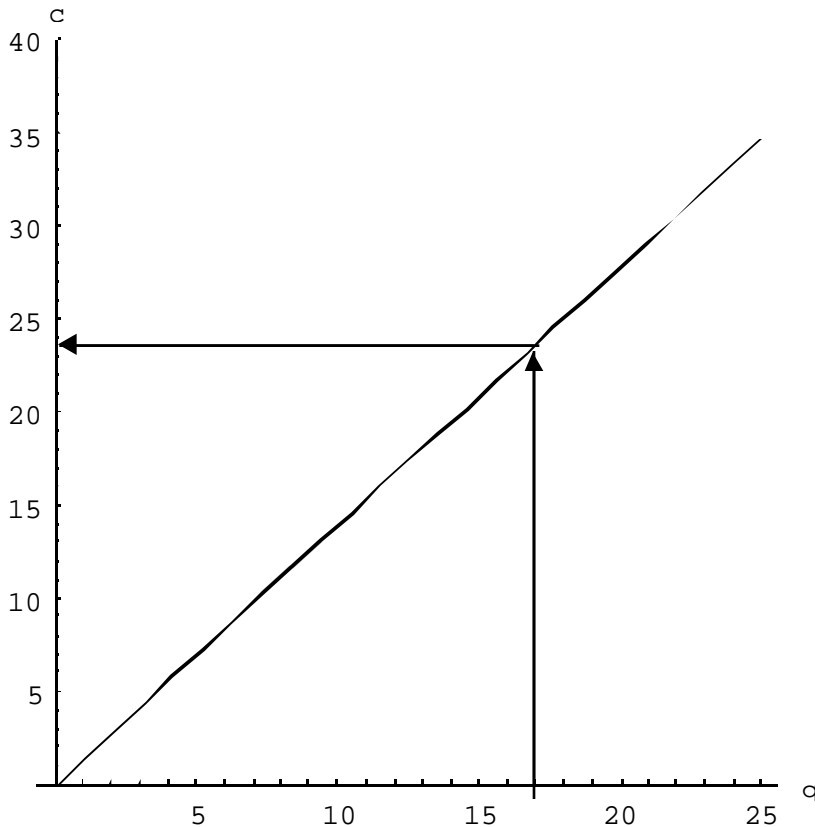
$$(0.719 \text{ gallons/dollars}) \cdot (20 \text{ dollars}) = 14.38 \text{ gallons}$$

This second problem could have also been solved by cross-multiplying, but using this approach would have involved the same sort of drawbacks as we saw using when cross multiplying in the first problem. Further, this approach would not have shown any relationship between the two problems. On the other hand, using the unit price and the unit amount, as we illustrated above, allows us to see clearly the connection between the two problems: namely, these unit amounts are reciprocals of each other:

$$\frac{1}{1.39} \quad 0.719 \text{ gallons per dollar} \quad \frac{1}{0.719} \quad 1.39 \text{ dollars per gallon}$$

That this is necessarily true follows from the very meaning of each of these quantities.

As a final remark we note that all problems of the type illustrated above can also be solved with a graph. In the case of problem 1 we would mark the point $(q, c) = (17, 23.63)$, and then draw a line through this point and $(0, 0)$. A line drawn up from $q = 17$ to the graph and then over to the vertical axis locates the answer $c = \$23.63$.



Note that the slope of this graph is just the unit price, \$1.39 per gallon, found above. This graph is the graph of a linear function, $c = 1.39 \cdot q$, that represents the general proportional relationship between cost and amount. The unit price method thus connects nicely with the work with algebra and functions that students will soon be immersed in. Another disadvantage of the cross multiplying method is that it is essentially a dead end that does not lead to representations in terms of graphs and functions.

For centuries the above sorts of problems have been called "rule of three" problems: three values are given and a fourth is sought. Cross-multiplying has been a method for solving such problems learned by many generations of school children. It arose in a context that valued rote methods applied without thought to solve certain standard problem types, and in an era before functions and their graphs were used at all in mathematics. It is surprising how persistent this method is even today given the change in times, the fundamental drawbacks of the method, and the arrival on the scene of functions and graphs as key algebraic concepts.

In addition, research (Hull, 2000) showed that when teachers set up problems similar to those above and found an answer by cross multiplying, a significant number had no idea how to interpret their answers because units were left out of the problem solving process. Moreover, units cannot be included since units found in the intermediate step in cross multiplying are meaningless. Another research study (Nunes, Schliemann, Carraher, 1993) showed that students can be taught cross multiplication, but then quickly forget the method and instead rely on other methods that make more sense to them when asked to solve problems in the real world.

Hull, S.H. (2000). *Teachers' mathematical understanding of proportionality: Links to curriculum, professional development, and support*. Unpublished doctoral dissertation, University of Texas, Austin.

Nunes, T., Schliemann, A.D., & Carraher, D.W. (1993). *Street mathematics and school mathematics* (pp. 77-126). New York: Cambridge University Press.