Symmetry of Design
Teacher Notes

The activities in this section review four rigid motions, translation, rotation, reflection, and glide reflection – and require that students identify these rigid motions in strip (or border) designs and finite designs. The students will also analyze strip designs to determine that only seven strip designs are possible from the various combinations of rigid transformations.

The first activity, Using Transformations to Create Designs, reviews three transformations translation, reflection, and rotation, and a fourth rigid motion, glide reflection, which is a combination of two transformations. It describes the transformation's elements and requires that the students perform several transformations on dot graphs. The elements of a transformation are the points, lines, direction, or units that are necessary to determine the motion of the object.

The second activity, Strip Designs, illustrates how the four rigid motions are found in strip (or border or frieze) designs. There are five rigid motions to look for because reflections are divided into those that are perpendicular to the midline of the design and those that are parallel to the midline. The midline of the design strip is a line parallel to the edges of the design and halfway between these edges. It also has students analyze strip designs to find the translation unit and, ultimately, the fundamental unit.

In Analyzing Strip Designs, the third activity in this section, students analyze strip designs in two other ways – by marking the elements of each transformation that exists in the design and by determining which combinations of transformations can possibly exist together. For example, a reflection parallel to the midline of the design and a reflection perpendicular to the midline cannot exist together without also having an 180° rotation (since consecutive reflections in perpendicular lines create a rotation).

The final activity, Finite Designs, has students examining finite designs in the same kinds of ways they examined strip designs. Finite designs are designs that do not have translation or glide reflection, only rotation and/or reflection. This activity establishes ideas that will be used in the next section on kaleidoscopes.

Required Materials

- student activity sheets, one per student
- blank transparency film and overhead pen (approximately one per student) or tracing paper (four or five sheets per student)
- ruler, one per student
- Transparencies
  
  Using Transformations to Create Designs Examples
  Examples of Strip Designs
  Possible Strip Design Combinations
  The Seven Strip Designs
  The Summary of Seven Types of Strip Designs
  Mandala Design
  Mandala
Optional Materials

- copy of transparencies, Examples of Strip Designs, for each student
- dot paper; isometric and rectangular grids
- transparencies of both isometric and rectangular grids
- colored pencils
- reflective surface such as a mirror or other reflector (like a Mira) to assist students in visualizing reflections
- examples of strip designs such as wallpaper borders, some quilt borders, photos of friezes in architecture, patterned ribbons, etc. (The Resources section lists World Wide Web sites with other examples.)
- examples of finite designs such as mandalas, other artwork (some Escher), logos, photos of car wheels or hubcaps, etc. (The Resources section lists World Wide Web sites with other examples.)

Vocabulary

**translation** - a translation is a shift or a slide of a figure or design in a given direction and distance.

**reflection** - a reflection is a mirror image of a figure or design across a line of reflection

**rotation** - a rotation is a turn of a figure or design around a specific point and specific angle.

**glide reflection** - a glide reflection is a combination of a translation and a reflection. The line of reflection is parallel to the direction of the translation.

**line of reflection** - the line that acts as the mirror; a point on the original figure and its image are equidistant from this line.

**rotational symmetry** - a figure has rotational symmetry if there is a point on the figure such that the figure can be turned around a certain number of degrees and still look the same.

**strip/border/frieze design** – a design that has translation symmetry in one direction only.

**translation unit** – the smallest portion of a strip design that can be translated to create the design.

**fundamental unit** – the smallest portion of a strip design that can be used to create the design using any of the transformations.

**finite/rosette design** – a design that does not have translation or glide reflection but has only rotations or reflections.

**midline of the design strip** - the line parallel to the edges of the design and halfway between these edges

Procedures

Each student activity begins with a "Notes" section. The teacher may ask the students to read this section first or use this as a guide to introduce the concept. Every student should have a piece of transparency film and overhead pens or tracing paper.
Using Transformations to Create Designs

Provide an overview of this section. Explain that the first activity is a review of transformations but later activities will have them analyzing existing designs to determine what transformations exist in the design as well to determine whether there are a finite or infinite number of designs possible based on the transformations. (Note: There are a finite number of combinations of transformations, but there are still an infinite number of transformations possible because translations can be of any length, rotations can have any center, with any number of degrees of turn, and reflections can use any line as a reflector.)

For each rigid motion:

1. Review the rigid motions described in the notes for each section; each definition is followed by an example in the student notes. These examples are shown on the transparency, Using Transformations to Create Designs Examples.
2. Lead them in a discussion of the properties of the transformations. Possible questions for discussion are bulleted in each section below.
3. Demonstrate for the students (using dot paper transparencies or the transparencies of the example problems from the activity), how they can use the blank piece of transparency to assist them in performing the transformations.
4. After each transformation is discussed allow time for students to work the problems on the activity sheet.

Translation

- What do you need to know in order to translate a figure?
- Describe how the original figure was translated.
- Explain how you know the number of units to move the figure.
- What is the location of the image of point C in the original figure?
- Compare the original figure and its image.

Show the students how to use the transparency film or tracing paper to create the image. Trace the slide arrow onto the transparency; line up the point end of the slide arrow with each vertex of the figure making sure that it is parallel to the original slide arrow to determine where that vertex is translated.

Reflection

- Describe how the point E’ was determined using the point E and the line of symmetry.
- Describe the relationship between the points D, D’ and the line of symmetry.
- Which points on the figures are three units from the line of symmetry?
- Compare the original figure and its image.

Model the use of the transparency or tracing paper to create the image. Trace the figure and line of reflection onto the transparency; flip the transparency over to mimic the reflection making sure the “lines” of reflection line up with each other; copy the reflected image onto the dot paper in the same location as the transparency image.

Rotation

- What information is given to help you find the new figure?
- Describe how the point A’ was determined using the point A and given elements of the rotation.
- Describe the relationship between the points A, A’, and A".
• How would the rotation have been different if you had rotate the figure once and A had the image point A''?  
• Compare the original figure and its image.

Model the use of the transparency film or tracing paper. Trace the turn arrow onto the transparency and relocate it to the center of rotation to aid in visualizing that every edge of the figure rotates the specified amount; trace the figure onto the transparency with the turn arrow still relocated to the center of rotation to determine where to copy the next image.

**Glide reflection**

• Look at the glide reflections that are shown with problem five. The shaded figure is the original figure. Describe the figure was created next if you performed a reflection and then a translation.
• How did you know how much to translate the figure?
• How was the next figure created?
• Compare the original figure to the figure above the line.
• What other transformation could be used to transform the original figure to get the figure above the line on the right?

Show the students how to use the transparency film or tracing paper. Trace the slide arrow and line of reflection onto the transparency; place the arrow end of the slide arrow on a vertex of the figure and trace the figure; now place the back end of the slide arrow on the same vertex and flip the transparency to mimic the reflection; copy the figure onto the new location on the dot paper

**Strip Designs**

In this activity you will first ask students to identify rigid motions on strip designs. They will next find translation units and fundamental units in the designs.

Show the transparency Examples of Strip Designs. Define a strip design as one that only has translation in one direction (horizontally or vertically but not both). Ask them to identify the transformations in several of the designs. You might ask a student to come to the overhead and circle the unit that was translated in the first two designs.

Ask the students to describe any lines of reflection in the first strip. (There are no lines of symmetry in the first strip.)

Ask the students if they can draw any lines of reflection in the second strip. (A vertical line of symmetry could be drawn between the first and second figures. There are more vertical lines of symmetry depending upon your choice of translation unit. There is no horizontal line of reflection.)

Ask the students if they can draw any lines of reflection in the third strip. (There is a horizontal line of reflection. The line is midway on the strip.)

Define the midline of the design strip to be the line parallel to the edges of the design and halfway between these edges. When students note that there are two types of reflections (reflections with the midline as the line of reflection and reflections with the perpendicular to the midline as the line of reflection), explain that they are going to consider and identify these as two transformations rather than just one. (This will prepare them for the Analyzing Strip Designs activity.)

Ask the students about the fourth strip.
• Does this strip have a line a symmetry? (no)
• What other transformations do you see in this strip? (The second figure in the strip could be obtained by a combination of a reflection and translations. The first figure is reflected across a horizontal line and shifted right and up.)

Ask the students about the fourth strip.
• Does this strip have a line a symmetry? (no)
• What other transformations do you see in this strip? (The second figure in the strip could be considered to be a 180-degree rotation of the first figure of the strip. The center of rotation is midway between the two figures.)
• How many degrees must the design be rotated until it returns to its original position?

(A 360° rotation is an identity rotation; it gets the design back to where it started. The preferred response is that the design is rotated 180° or a half-turn. For many objects 180° is the minimum rotation that allows the design to look the same as it did originally.)

Discuss the rest of the examples, listing which of the five rigid motions (translation, half-turn rotation, reflection about the midline, reflection perpendicular to the midline, and glide reflection) exist in each.
6. reflection over midline, reflection across perpendicular to the midline, rotation, translation
7. translation
8. rotation, translation
9. rotation, translation
10. rotation, translation
11. reflection over perpendicular to the midline, rotation, translation

Distribute the activity Strip Designs. The Notes section is a basis for class discussion. Use this information to define and describe a translation unit. Ask the students to find the translation unit of several of the examples on the Examples of Strip Designs transparencies in addition to those example problems in the Notes.

Consider the pottery design on the second page of the notes to model the difference between the translation unit and the fundamental unit. Using the examples in the Notes section of the activity, or Examples of Strip Designs ask the students to draw a rectangle around the translation unit of several of the designs.

Show page one of the Examples of Strip Designs Transparency and ask them to find an example where the fundamental unit is not the translation unit. (The second strip design has a reflection and translation. The first quadrilateral is the fundamental unit. The first two quadrilaterals form the translation unit.

Ask someone who has seen that there could be a smaller unit to describe that smaller unit that will generate the translation unit and how it could be created. (See the student activity for more detail on the fundamental unit.) Point out, using the examples in the activity, that the translation unit could be a reflection, rotation, or glide reflection of a smaller unit called the fundamental unit. Using the examples where students drew a box around the translation unit, ask them to find the fundamental unit for those designs. The teacher may also give students transparencies of a design and the design on the paper so that the students can cut the transparency and move around on the paper copy of the design to find the congruent shapes.

Before students begin working on the problems, clarify whether the fundamental unit should be boxed or shaded. Students might be more comfortable finding both, so one could be boxed and the
other shaded. Allow them to work on the activity individually or in small groups of 2 or 3 students while you monitor and respond to questions. Allow the students time to work on the first part of the activity, Problems 1-6.

Discuss each problem and the combinations of transformations that are used in the strip design.

Allow time to work on problem 7.

### Analyzing Strip Designs

In this activity, students will identify the transformations and indicate all unique lines of reflection and centers of rotation. Provide several examples, using the Examples of Strip Designs transparencies or other preferred examples, of marking unique lines of reflection and centers of rotation, as well as translations and glide reflections. Lines of reflection could be identified with dotted lines and centers of rotation with a dot or small square. Glide reflections could be identified by circling the portions of the glide reflection with the reflection marked with a solid line. Teachers should develop a system and relay this information to their students.

Allow the students time to work on the first part of the activity, Problems 1-7.

Discuss each problem and the combinations of transformations that are used in the strip design. Write these combinations on the board or on a transparency.

1. rotation, translation
2. reflection perpendicular to the midline, translation
3. reflection perpendicular to the midline, reflection along the midline, rotation, translation
4. reflection perpendicular to the midline, translation
5. reflection perpendicular to the midline, rotation, translation
6. reflection perpendicular to the midline, reflection along the midline, rotation, translation
7. glide reflection, translation

The next part of the activity leads students to the conclusion that there are only seven possible strip designs based on the transformations. Prior to going on with the activity, lead the students in a discussion on the variety of combinations of transformations they have seen in the strip design examples and the problems. Refer to the list of combinations on a transparency or the chalkboard that they remember seeing (without looking ahead in the activity). Then ask:

- Are there any combinations of transformations that you can think of that we have not listed here?

There may be one or two additions to the list. Then pose the following question:

- Without looking ahead, how many combinations of transformations ARE there?

The accurate response may not be obvious or forthcoming. The rationale for why there are sixteen combinations: since translation is required for a design to be a strip design, there are four remaining transformations. Each one can either exist in the design or not exist in the design, 2 options. Through counting techniques (combinatorics), it can be determined that there are \((2)(2)(2)(2) = 16\) possible combinations of the four transformations. At this point, have the students refer to the table of possibilities and see which ones may have been left off the list on the board. Explain that this is a list of possible combinations, it is reasonable that not all of these combinations can actually exist together. Help them get started with the next homework problem by reiterating the text of the activity and clarifying the task. Possibly suggest that they go through this and the previous
activities and indicate on the table which ones they are sure exist because there are examples of these combinations. This will limit the cases that they actually have to analyze themselves. Detailed explanations of why cases do not exist are provided with the answers.

Allow the students to work individually or in small groups.

After the students have completed the activity, the transparencies The Seven Strip Designs may be shown to the students to assist them in visualizing the seven strip designs that exist.

The Summary of Seven Types of Strip Designs can also be used to reinforce translation units and fundamental units, as well as the seven strip designs.

**Finite Designs**

This activity revisits the idea of a fundamental unit but using designs that have no translation or glide reflection.

Define and describe a finite design. Show the students the examples gathered or the transparencies, Mandala Design and Mandala. Ask the students to analyze the examples given in the activity by drawing in lines of reflection, marking the center of rotation, and determining the number of degrees (or the \( n \)-fold description*) in the rotation. Go over these examples with them after they have had several minutes to complete their analysis.

Once satisfied that students understand the properties of finite designs and marking the transformations, allow them to work on the activity individually or in small groups of 2 or 3 students.

**Symmetry Project**

Distribute the Symmetry Project and allow students time to work on it in class, if extra time is available. The Symmetry Project is the major assessment for this section of the Geometry Unit.

In the pilot of these materials, this unit occurred at the end of a spring semester. In lieu of an exam, students were asked to complete the Symmetry Project to assess their knowledge and understanding of the concepts. This project counted as one-third of a regular exam grade. The included project does not cover finite designs but similar tasks on finite designs could be included.

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* The \( n \)-fold description of rotations utilizes the fact that \( n \) copies of the fundamental unit are required to complete the entire design. The floral mandala example in the student activity has seven-fold rotation since seven copies of the fundamental unit are required to complete the design.
Symmetry of Design Resources

In addition to the transparency masters and other resources that follow, the following resources may be useful or informative:


(Unlike other references mentioned here, this is an in-depth analysis of design classifications using advanced mathematics.)


KaleidoMania!™ Interactive Symmetry, Kevin D. Lee, Sandpiper Software
*KaleidoMania!* is a unique new tool for dynamically creating and analyzing symmetric designs and for exploring the mathematics of symmetry. This new CD-ROM software and its accompanying book of blackline masters, the *KaleidoMania! Interactive Symmetry Activity Book*, by Lois Edwards, offer a comprehensive, interactive unit on transformational geometry and symmetry.


The following websites on strip designs may be useful or informative:

http://mathforum.org/geometry/rugs/symmetry/
This site contains information about transformations and definitions, with examples, of the four transformations. It also contains a gallery of rugs that could be used as examples for analysis.

www.fandm.edu/departments/mathematics/a.crannell/hm/Math/frieze.html
This site defines the four transformations and provides examples of the seven classifications of frieze patterns.

www.geom.umn.edu/~lori/kali
This site defines the transformations and has examples of strip designs. It also discusses using the software program Kali to explore strip designs. (Kali is available free of charge from www.geom.umn.edu/software/download/kali.html courtesy of the Geometry Center at the University of Minnesota.)

www.math.okstate.edu/~rpsc
This is a nice site that has lots of information about the transformations of decorative art which includes mandalas and border designs. The worksheets have lots of examples of both finite and strip designs. Some of the student examples were created by high school students during a summer program.

www.math.okstate.edu/~wolfe/border/border.html
This site has examples of strip designs created by students at Oklahoma State University.

www.math.uic.edu/~burgiel/Mht420/4/activity.html
This site has examples of the seven type of strip designs.

www.nrich.maths.org/mathsfl/journalf/nov98/art1/
This site has examples of strip designs found in cast ironwork in Sydney, Australia.

www2.austin.cc.tx.us/hannigan/Presentations/NSFMar1398/MathofSP.html
This site defines the four transformations and provides examples of Seminole Patchwork designs that are strip designs. It also has additional information on Seminole designs and traditions.

The following websites on finite designs may be useful or informative:

members.aol.com/rcmoeur/signman.html
Manual of Traffic Signs – This is a source of finite designs that students see regularly. Students can analyze these for symmetry.

www.abgoodwin.com/mandala/
This site has examples of mandalas as well information on mandalas and their cultural significance. It also offers software, Mandala Maker, for sale through the website.

www.earthmandalas.com
This is a commercial site selling mandalas made from photographs of nature. The mandalas are very pretty but may be difficult for students to analyze.

www.earthmeasure.com
This site contains a information about Native American Geometry and a collection of designs that could be used as examples of finite designs.

www.folkart.com/~latitude/home/hex.htm
This site is a commercial site offering Hex Signs (created by the Pennsylvania Dutch). The designs on the site could be used as examples of finite designs.

www.japan-society.org/crest_jssdt.html
This site has a few examples of Japanese family crest that could be used as examples of finite designs.

www.mandali.com
This site is a commercial site offering coloring books of mandala designs. There is also some information about mandalas and the artist. The books are $8.95 - $9.95 and contain black and white examples of finite designs.

Acknowledgement

The mandala examples in this section were used from *Everyone’s Mandala Coloring Book* by Monique Mandali. We would like to thank Monique for giving us permission to use her designs.
This, and other books, are available from Mandali Publishing, PO Box 219, Helena, Montana 59624. Phone 800-347-1223; web www.mandali.com.
Using Transformations to Create Designs

Examples

Translation

Reflection
Rotation

Glide Reflection
Examples of Strip Designs

1

2

3

4

5

6
## Possible Strip Design Combinations

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<th>Translation</th>
<th>Reflection Perpendicular to the Midline</th>
<th>Reflection about the Midline</th>
<th>Rotation</th>
<th>Glide Reflection</th>
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The Seven Strip Designs

- Translation only
- Translation with reflection perpendicular to the midline
- Translation with rotation
- Translation with reflection about the midline
The Seven Strip Designs

- Translation with glide reflection

- Translation with reflection perpendicular to the midline, rotation, and glide reflection

- Translation with reflection perpendicular to the midline, reflection about the midline, and rotation
Summary of Seven Types of Strip Designs

For each of the designs below the translation unit is circled and the lines of reflection and the points of rotation are marked.

The fundamental unit is copied below the design strip.

1. Translation

![Translation Design]

Fundamental unit:

2. Translation with reflection parallel to the midline

![Translation with Reflection Design]

Fundamental unit:
3. Translation with reflection perpendicular to the midline

![Diagram of translation with reflection]

Fundamental unit:

4. Translation with rotation

![Diagram of translation with rotation]

Fundamental unit:
5. Translation with glide reflection

Fundamental unit:

6. Translation with reflection perpendicular to the midline, rotation, and glide reflection

Fundamental unit:
7. Translation with reflection about the midline, reflection perpendicular to the midline, and rotation
Mandala Design

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Mandala

Reproduced with permission of the artist Monique Mandali.
Using Transformations to Create Designs

Notes

In geometry, there are four ways to “change” or “transform” a geometric figure without changing its shape. These are called rigid transformations or rigid motions. When we discuss these transformations in the context of designs, we refer to them as transformations because they are repetitions of a single figure or design. What we’re looking for is a figure or design that moves “onto” itself in such a way that it exactly congruent to the pattern in the original position.

We can apply these four rigid motions to designs; both to use simple figures to create a design using the transformations and also to classify designs based on the transformations that already exist in the design.

Translation

One of the transformations is a translation; a translation is just a shift or a slide of a figure or design. There are two elements of a translation – the direction in which it shifts and the distance the figure is shifted. These two elements are generally described by using a slide arrow. The direction the slide arrow points tells you which direction to shift the figure. The length of the slide arrow tells you the distance the figure needs to be shifted.

In the example below left, the trapezoid is the figure to be translated. The slide arrow tells you that the trapezoid needs to be shifted down and right at a 45° angle for a distance of “one diagonal” (the diagonal distance between two dots). In the example on the right, the same trapezoid has been shifted one time. As a reference, each vertex of the original trapezoid was moved “one diagonal” down and right.
In the example below, the shaded triangle has been translated up and right by a distance equivalent to the diagonal of a 1x2 rectangle. Notice that there are two translated copies of the triangle; the same translation (direction and distance) has been completed twice using the triangle created after the first translation as the starting point for the second translation.

Problem:

1. Translate the following figures using the slide arrows given. Draw at least two copies of the original figure.
Reflection

A second transformation is the reflection; a reflection is a mirror image of a figure or design (also referred to as a flip). There is only one element of a reflection and that is the line of reflection or the line of symmetry. It is the line that acts as the mirror; a point on the original figure and its image are the same distance from the line of reflection, but on opposite sides of this line.

In the example below, the line of reflection is the dotted line. The figure on the left is the original figure that has been reflected over the dotted line to create the figure on the right.

In the example on the next page, the figure on the right is the original figure. Since it is touching the line of reflection, the image is also touching
the line of reflection, as well as the original figure, making the two figures look like one.

Problem:

2. Draw the reflection of each of the following images using the given line of reflection.
Rotation

The third rigid motion is a rotation; a rotation is a turn of a figure or design around a specific point. There are three elements of a rotation – the direction of the turn (clockwise or counter-clockwise), the angle of rotation, and the center of rotation. The direction and angle are often described using a turn arrow. The center is a point about which a figure or design is rotated. To determine the image of a point on the original object a line segment is drawn from the center of rotation to the point and the line is rotated through the given angle in the direction of the turn with the center of rotation as the vertex of the angle.

In the example on the left, the center of rotation is the darkened point at the bottom vertex of the triangle. The turn arrow tells you to rotate the triangle 90° counter-clockwise. In the example on the right, the triangle has been rotated three times, using the first triangle created as the starting point for the second, the second as the starting point for the third, etc. Notice that every line in the triangle has been rotated 90°.

This next example shows that the center of rotation does not have to be a vertex of the figure; it could be located anywhere. The original (shaded) trapezoid is rotated 120° clockwise and the rotation is completed twice to get a “complete” design. (A third rotation would put a new trapezoid directly on top of the original one.)
Problem:
3. Draw at least two rotations of each of the following figures using the turn arrows and centers of rotation that are given.
4. All of the designs you created using rotations eventually repeated themselves (were eventually cyclic). The angles of rotation you used were $45, 60, 90, 120,$ and $180$ degrees. What angles of rotation would create designs that are NOT cyclic within $360$ degrees of rotation?
**Glide Reflection**

The fourth rigid motion is the **glide reflection**; a glide reflection is a combination of a translation and a reflection. The line of reflection is parallel to the direction of the translation. There are three elements to a glide reflection: the direction of the translation, the distance of the translation, and the line of reflection.

The best example of a glide reflection is footprints. Your feet are reflections of each and when you walk, each foot slides forward a certain amount. The distance of the translation is the length of your stride (from left toes to right toes).

In this example, the distance and direction of the translation is shown by the slide arrow, the dotted line is the line of reflection. The glide reflection has been completed five times.

**Problems:**

5. Add at least three more images to this glide reflection.
6. Draw a glide reflection of each of the following images using the given line of reflection and slide arrow. Draw at least five images beyond the original.

(a)

(b)

7. List the four rigid transformations and, in your own words, describe the elements of each.
Strip Designs

Notes

In this project, we are going to look at strip designs. You’ve probably seen lots of strip designs, also called border designs) and not even realized it. Wallpaper borders, some decorative ribbon, certain kinds of lace, friezes in architecture - strip designs are everywhere.

When a design has translation, it can be repeated an infinite number of times. Strip designs are those designs that have translations along a single line. This single line could be horizontal or vertical; most of the strip designs in this activity have translation along a horizontal line.

To create a strip design, you need a translation unit. This translation unit is what will be infinitely repeated along the strip.

When you look at this Japanese design, you can see that flower with stems and leaves repeats and can keep repeating an infinite number of times. Some people will see the translation unit as puzzle pieces where each complete flower and stem design stays intact.

While this is perfectly reasonable, mathematically we want to think of the translation unit as a rectangle. When you think of the translation unit as a rectangle, you might consider either of the rectangles shown below.
Either way you decide to create the rectangles, notice that the rectangle that creates the translation unit is the same size. In finding this rectangle it may be helpful to think of mass-producing a wallpaper border. Wallpaper manufacturers put a limited portion of the design, the translation unit, on a printing cylinder. This cylinder then rolls over the blank wallpaper creating the wallpaper design. So, you want to think “what’s the smallest section of the whole design that is necessary to put on the printing cylinder?”

The translation unit may be the smallest design element, but sometimes it is not. Because of the effect the artist is trying to get, the whole strip design may contain additional transformations that are not necessarily apparent in just the translation unit.

Look at the pottery design at the right. There is a line of reflection in the design running down the middle of the strip. If you folded the strip in half (the left edge matching the right edge), the design would lie exactly, barring slight imperfection, on top of itself. We’ll call this an example of reflection along the midline. The midline of the design strip is the line parallel to the edges of the design and halfway between these edges.

The translation unit for this design would be one of the figures, as shown below on the left. However, the most basic design unit, the fundamental unit, is only half of the translation unit as shown below on the right.
What may be the most obvious rigid transformation in the next Japanese design is the glide reflection or the lines of reflection that run perpendicular to the midline.

Probably the least obvious transformation is the rotation. You can tell if a strip design has rotation if you can turn the design upside down and it still looks the same. Because the strip has to retain its original orientation, in this case horizontal, the rotation angle can only be 180°. You have to be careful to look for rotation within the entire strip design not in one piece of the design.

The translation unit for this design must include two of the “points”. To create a rectangle, the upper “point” has been cut in half, but there is still one upper “point” and one (complete) lower “point.” (Again, there are slight imperfections in the design that you must ignore.)

This translation unit is not the smallest possible design unit, however. The line of reflection that is perpendicular to the midline allows us to divide the translation unit into a smaller unit (see below left). If you look carefully, this portion of the design can be rotated about the center of the rectangle. We can use this to divide the unit into a smaller one, the fundamental unit (see below right).
To create the entire strip design, the fundamental unit can be rotated 180° about the midpoint of the top edge (so that the copy of the design is on top of the fundamental unit), then reflected about either of the left or right edges, and then translated an infinite number of times.

This last example was nice one because the rotation was readily apparent and in the center of the translation unit; this isn’t always the case. In this Japanese design, a seemingly obvious place to cut out the translation unit doesn’t leave the rotation in the center of the translation unit.

In this example, there are two possible centers of rotation, both along the top edge of the fundamental unit rectangle. Either would create an odd-shaped translation unit; while not preferred, this can happen.
If you want to have a “nice”, rectangular translation unit, make sure the center of the rotation is in the center of your initial translation unit.

Let’s look at one more example. This Indian design has a line of reflection along the midline, lines of reflection perpendicular to the midline, and rotation in addition to translation.

The translation unit would be one pair of top/bottom flowers. The line of reflection perpendicular to the midline would divide that in half vertically and the line of reflection along the midline would divide it in half again horizontally. In this case there is no need to worry about the rotation in finding the fundamental unit. There are two possible centers of rotation at the top corners of the fundamental unit but applying reflections eliminates the need to apply the rotation in creating the strip design.
Problems:

For each of the following strip designs, list the rigid motions that are present in the design (translation, rotation, glide reflection, reflection along the midline, reflection perpendicular to the midline) and find a fundamental unit of the design. Indicate the fundamental unit by shading it in or drawing a rectangle around it.

These strip designs are recreations of traditional Seminole Indian designs.

1. Alligator Tracks

2. Beaver Teeth

3. Whirlwind
4. Wolf Fangs

5. Raccoon Tracks

6. Four Crossed Logs
7. Color each of the following designs so that it has ONLY the transformations listed.

- Translation and reflection along the midline

- Translation and reflection perpendicular to the midline

- Translation and glide reflection

- Translation only
Analyzing Strip Designs

Notes

Finding the fundamental unit of the design is one way to analyze strip designs. There are two other ways to analyze strip designs that we’ll cover in this activity.

This next way to analyze designs involves finding all unique lines of reflection and centers of rotation. If a strip design has reflection along the midline there is only one possible line of reflection, so this is the easy case. However, if a strip design has reflection perpendicular to the midline there might be more than one unique line of reflection. Similarly, there may be more than one point that could be the center of rotation for the strip design. “Unique” means that the lines of reflection or centers of rotation are in different positions relative to the design.

In this Japanese strip design, there is a line of reflection perpendicular to the midline through the middle of the “tree.” Even though there are several lines of reflection, each one occurs in the same place in the design, the middle of the “tree.” The vertical white lines in the second picture are the lines of reflection.

![Design Example]

There are also lines of reflection between each tree. This is a different position in the design so it is considered a second unique line of reflection. So, this strip design has two unique lines of reflection perpendicular to the midline. The thicker white line indicates this second unique reflection.
Examine another Japanese strip design.

It’s clear that this design has rotation because if you turn it 180° it looks the same as it does now. It may not be so clear where the centers of rotation are. First, the centers of rotation must always lie on the midline (note that the midline is not acting as a line of reflection in this case but there is still a midline); if the strip is going to keep the same orientation as it does now, it has to rotate about some point on the midline. In this design there are centers of rotation in between where the flowers arc together as indicated by the white square on the next picture.

Even though three centers of rotation are marked, they are only one unique center because they are all in the same position relative to the strip design. There is another unique center of rotation; this is located between the flowers on the other side, where they are flatter. These centers are indicated by a white circle on the following picture.
This next design has two unique centers of rotation and two unique lines of reflection perpendicular to the midline. Can you locate them without looking at the second picture?

Since the downward opening flowers are different from the upward opening flowers (even though they are a glide reflection of each other), there is a unique line of reflection through the middle of each "kind" of flower. There is a center of rotation in between leaves and since the left leaf is a different position in the design than the right leaf, each kind is a different center of rotation. In the next picture, the lines of reflection are marked with thick and thin white lines and the centers of rotation are marked with white squares and triangles.
Problems:

Find and mark the unique lines of reflection (including any reflections along the midline) and unique centers of rotation in each of the following strip designs.

1. 

2. 

3. 
4. [Image of a strip design]

5. [Image of another strip design]

6. [Image of a third strip design]

7. Whirlwind [Image of a fourth strip design]
Now let's examine another way to analyze strip designs. As you may have noticed while looking at different strip design up to this point, not every design has every transformation and some combinations of transformations seem to occur together frequently. Since there are only the five transformations to consider (we'll consider reflection along the midline and reflection perpendicular to the midline to be different transformations), we can make a chart that lists all possible combinations of transformations.

Since every strip design must have translation to be a strip design there are only four transformations that have to be counted when deciding how many combinations there are. There are two options – the strip design has a transformation or it doesn’t so there are \((2)(2)(2)(2)=16\) possible combinations of the five transformations. The table on the next page lists all those possible combinations.

Some of these combinations can “exist” together, others cannot; this table presents all the possible combinations, some of these are not possible to create. Note that if three rigid motions are marked, for example in Row 6 the combination is translation, reflection perpendicular to the midline, and reflection along the midline, this means that those three transformations and ONLY those transformations must be present in the strip design.

8. Your next task is to decide which of the possible combinations of transformations really do exist. In the last column of the chart mark the combination as P (possible) or NP (not possible). For each combination, explain why the combination does or does not exist. Use both verbal explanations and examples of strip designs (use simple ones) in your explanation.

For example:
Row 1: Possible combination. A strip design can have only translation. There was an example of this in the last project, the Japanese flower. I can also draw an example using right triangles:

```
\[\begin{array}{c}
\text{ translation } \\
\text{ translation } \\
\text{ translation } \\
\text{ translation }
\end{array}\]
```

*Since the triangle itself has no symmetry, there is only the transformation of translation in the strip design.*

Now, you provide explanations for the remaining combinations. (Hint: there are only 7 combinations of transformations that really exist in strip designs.)
<table>
<thead>
<tr>
<th></th>
<th>Translation</th>
<th>Reflection Perpendicular to the Midline</th>
<th>Reflection Parallel to the Midline</th>
<th>Rotation</th>
<th>Glide Reflection</th>
<th>Possible (P)/ Not Possible (NP)</th>
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Finite Designs

In this activity, we’ll be analyzing designs again but designs that were not created by translation or glide reflection. These designs, like so many others, both natural and man-made, have a repeating element that creates the design but the design is created only through rotations and reflections.

The simplest type of design is a finite design, also called a rosette design. In a finite design, the repeating element repeats through reflection or rotation but never by translation or glide reflection. Examples of finite design are the rotation problems from the Using Transformation to Create of Design assignment, many flowers, and pinwheels.

The design below is called a mandala. Mandalas are generally considered to be spiritual icons and are found in many cultures.

Reproduced with permission of the artist Monique Mandali.

It is an example of a finite design that has 7-fold rotational transformation. We refer to it as 7-fold because the repeating element (also called the fundamental unit) appears seven times. If you draw a line from the center of the design through each tip of the ribbon (going between the flower-like designs) you will have created the fundamental unit and the six repetitions.

The lines that define the boundary of the fundamental unit don’t need to go through the tips of the ribbon, they could go anywhere as long they divide the circular design in to seven congruent parts.
There are two kinds of symmetry. A figure has **line symmetry** if there is a line in the figure, such that a portion of the design has been reflected. A figure has **rotational symmetry** if there is a point on the figure such that the figure can be turned around a certain number of degrees and still look the same.

Designs may have one or both of these symmetries. Describe the symmetry in this figure.

This design has a line of reflection but does not have rotational symmetry. Some designs are based on both rotational and reflection. The example of a mandala on the next page involves both rotation and reflection. Identify the fundamental unit (the most basic repeating element of the design) and draw in the lines of symmetry.
Did you find there are 16 lines of symmetry and that there are 16 copies of the fundamental unit? Yes, 16 copies of the fundamental unit. Even though this design only has 8-fold rotational symmetry, the fundamental unit is only half of that because of the reflection symmetry.

We describe this mandala design as having 8-fold rotational symmetry; it has eight lines of symmetry.

Other examples of finite designs are company logos. Many companies and corporations have design logos (versus stylized lettering logos) that can be described using transformations. Some readily identifiable ones are Texaco, Shell, and Mitsubishi. See how many others you can think of.
Problems:
Draw any lines of symmetry and find the fundamental unit for each of the following finite designs. Lightly shade in the fundamental unit. Describe the design using its symmetries (x-fold rotation, y lines of symmetry). Ignore any slight imperfections in the designs.

1.

2.

3.

4.
Symmetry Project

In the activities that we have been doing the last several days, you have learned to analyze strip designs and you have seen that there are seven possible design patterns for border patterns.

The seven types of strip designs are:

1. Translation only
2. Reflection along the midline and translation
3. Reflection perpendicular to the midline and translation
4. Rotation and translation
5. Glide reflection and translation
6. Reflection perpendicular to the midline, glide reflection, rotation, and translation
7. Reflection perpendicular to the midline, reflection across the midline, rotation, and translation

You are to create four border designs using four of the different types of strip designs (use four of the seven types) and you must meet the following requirements for each design:

1. draw each design so that it contains at least three translation units
2. identify the existing symmetries (in writing)
3. identify and draw any unique lines of symmetry
4. identify and draw any unique points of rotation
5. identify a translation unit
6. identify the fundamental unit

Grading Rubric

Draw four different designs using different types 48 points __________
Identify the transformations (in writing) of the seven designs 8 points __________
Identify the lines of symmetry 8 points __________
Identify the points of rotation 8 points __________
Identify the translation unit 8 points __________
Identify the fundamental unit 8 points __________
Extra quality points 12 points __________

100 points
Translate

1.

(a)

(b)
Reflection
2.

(a)

(b)
(c) 

(d) 

(e)
Rotation
3.

(a) 

(b) 

(c)
4. All of the designs you created using rotations eventually repeated themselves (were eventually cyclic). The angles of rotation you used were 45, 60, 90, 120, and 180 degrees. What angles of rotation would create designs that are NOT cyclic?

Sample response: Angles such as 70 degrees (not a divisor of 360) would produce designs that do not have rotational symmetry. Another example that would work is 22.5 degrees. If you rotate the figure 16 times, the figure would be in the original position.

Glide Reflections

5.
6. 

(a) 

(b) 

7. 

*Answers will vary but should mention the elements of the transformation.*
Strip Designs Answers

1. Alligator Tracks
   \[\text{translation, rotation}\]

2. Beaver Teeth
   \[\text{translation}\]

3. Whirlwind
   \[\text{translation, glide reflection}\]
4. Wolf Fangs

*translation, reflection perpendicular to the midline*

5. Raccoon Tracks

*translation, reflection along the midline*

6. Four Crossed Logs

*translation, reflection perpendicular to the midline, reflection along the midline, rotation*
7. *Answers may vary. Possible answers:*

- Translation and reflection along the midline
- Translation and reflection perpendicular to the midline
- Translation and glide reflection
- Translation only
Analyzing Strip Designs

Answers

1. Possible centers of rotation

2. 1

Possible centers of rotation
3. Centers of Rotation

Three lines of symmetry and two centers of rotation.

4.

5.

Center of Rotation
6. Center of rotations

Three lines of symmetry and two centers of rotation.

7. Whirlwind

No lines of symmetry or centers of rotation.
8. For example:

Row 1: Possible combination. A strip design can have only translation. There was an example of this in the last project, the Japanese flower. I can also draw an example using right triangles:

```
\[
\begin{array}{cccc}
\text{Translation} & \text{Reflection Perpendicular to the Midline} & \text{Reflection Parallel to the Midline} & \text{Rotation} & \text{Glide Reflection} & \text{Possible (P)/Not Possible (NP)} \\
1 & X & & & & P \\
2 & X & X & & & P \\
3 & X & & X & & P \\
4 & X & & X & & P \\
5 & X & X & X & & P \\
6 & X & X & X & & NP \\
7 & X & X & X & X & NP \\
8 & X & X & & X & NP \\
9 & X & & X & X & NP \\
10 & X & & X & X & NP \\
11 & X & & & X & NP \\
12 & X & X & X & X & P \\
13 & X & X & X & & NP \\
14 & X & X & & X & NP \\
15 & X & & X & X & NP \\
16 & X & X & X & X & NP \\
\end{array}
\]
```

Comments on those that are not possible:

Row 6: Not possible. If a design has two intersecting lines of reflection, then it must also have rotation. (A rotation can be defined as the image after reflections through intersecting lines.) This combination cannot exist since rotation must exist if there are two lines of reflection.

Row 7: Not possible. A similar argument as for the Row 6 combination. If a rotation and one line of reflection exist, there must also be a second line of reflection.

Row 8: Not possible. If there is a reflection perpendicular to the midline then there cannot be a glide reflection. These two transformations cannot exist together in a strip pattern.
Row 9: Not possible. Same argument as for Row 7. If a rotation and one line of reflection exist, there must also be a second line of reflection.

Row 10: Not possible. If there is a reflection along the parallel then there cannot be a glide reflection. These two transformations cannot exist together in a strip pattern.

Row 11: Not possible. Since a glide reflection has reflection in it, a single reflection and a rotation cannot appear together. There must be two lines of reflection or none when there is a rotation in the design.

Row 13: Not possible. If two lines of reflection exist then a rotation must also exist.

Row 14: Not possible. Same argument as for Row 7. If a rotation and one line of reflection exist, there must also be a second line of reflection.

Row 15: Not possible. Same argument as for Row 7. If a rotation and one line of reflection exist, there must also be a second line of reflection.

Row 16: Not possible. A glide reflection cannot exist if there are two lines of reflection and a rotation. It might appear that one could be identified, but the rotation takes precedence as a simpler transformation. In addition, a reflection along the midline and a glide reflection cannot both exist in the same strip pattern.
Finite Designs
Answers

1. 4-fold rotation, no lines of symmetry
    1 line of symmetry

2.

3. 6-fold rotation, 6 lines of symmetry

4. 4-fold rotation, 4 lines of symmetry
5. 4-fold rotation

6. 3-fold rotation, 3 lines of symmetry